

# On the Rayleigh–Taylor instability for the incompressible viscous magnetohydrodynamic equations

Fei Jiang<sup>a,\*</sup>, Song Jiang<sup>a</sup>, Yanjin Wang<sup>b</sup>

<sup>a</sup>*Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China.*

<sup>b</sup>*School of Mathematical Sciences, Xiamen University, Fujian 361005, China.*

---

## Abstract

We study the Rayleigh–Taylor instability problem for two incompressible, immiscible, viscous magnetohydrodynamic (MHD) flows with zero resistivity and surface tension (or without surface tension), evolving with a free interface under presence of a uniform gravitational field. First, we reformulate the MHD free boundary problem in a infinite slab as a Navier–Stokes system in Lagrangian coordinates with a force term induced by the fluid flow map. Then, we analyze the linearized problem around the steady state which describes a denser immiscible fluid lying above a light one with a free interface separating the two fluids, and both fluids being in (unstable) equilibrium. By studying a family of modified variational problems, we construct smooth (when restricted to each fluid domain) solutions to the linearized problem that grow exponentially fast in time in Sobolev spaces, thus leading to an global instability result for the linearized problem. Finally, using these pathological solutions, we prove the global instability for the corresponding nonlinear problem in an appropriate sense. Moreover, we evaluate that the so-called critical number indeed is equal to  $\sqrt{g[\varrho]/2}$ , and analyze the effect of viscosity and surface tension on the instability.

*Keywords:* Rayleigh–Taylor instability, MHD, free boundary problem, variational method.

*2000 MSC:* 76E25, 76E17, 76W05, 35Q35.

---

## 1. Introduction

Considering two completely plane-parallel layers of immiscible fluids, the heavier on top of the lighter one and both subject to the earth’s gravity. In this case, the equilibrium state is unstable to sustain small perturbations or disturbances, and this unstable disturbance will grow and lead to a release of potential energy, as the heavier fluid moves down under the (effective) gravitational force, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh [13, 14] and then Taylor [16], and is called therefore the Rayleigh–Taylor (R–T) instability. In the last decades, this phenomenon has been extensively investigated from both physical and numerical aspects and there are a lot of related results in the literature. In particular, many results concerning the linearized problems have been summarized in monographs, see, for instance, [2, 17]. To our best knowledge, however, there are only few mathematical analysis results on nonlinear problems in the literature, due to the fact that in general, passage from a linearized instability to a dynamical nonlinear instability for a conservative nonlinear partial differential system is rather difficult.

---

\*Corresponding author: Tel +86 15001201710.

*Email addresses:* jiangfei0591@163.com (Fei Jiang), jiang@iapcm.ac.cn (Song Jiang), wangyanjin\_2008@163.com (Yanjin Wang)

The magnetohydrodynamic (MHD) analogue of the R-T instability arises when the fluids are electrically conducting and a magnetic field is present, and the growth of the instability will be influenced by the magnetic field due to the generated electromagnetic induction and the Lorentz force. This has been analyzed from the physical point of view in many monographs, see, for example, [2, 17]. Because of additional difficulties induced by presence of the magnetic field, many results concerning the R-T instability of superposed flows could not be directly generalized to the case of MHD flows.

In this paper, we study the R-T problem for two incompressible, immiscible, viscous magnetohydrodynamic (MHD) flows, with zero resistivity and surface tension (or without surface tension), evolving with a free interface in the presence of a uniform gravitational field. We will prove that in Lagrangian coordinates, the corresponding MHD linearized system is globally unstable in some sense as time increases, and moreover, the original nonlinear problem with or without surface tension is globally unstable in some appropriate sense. Next, we formulate our problem in details.

### 1.1. Formulation in Eulerian coordinates

We consider the two-fluids free boundary problem for the equations of magnetohydrodynamics (MHD) within the infinite slab  $\Omega = \mathbb{R}^2 \times (-1, 1) \subset \mathbb{R}^3$  and for time  $t \geq 0$ . The fluids are separated by a moving free interface  $\Sigma(t)$  that extends to infinity in every horizontal direction. The interface divides  $\Omega$  into two time-dependent, disjoint, open subsets  $\Omega_{\pm}(t)$  so that  $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \Sigma(t)$  and  $\Sigma(t) = \bar{\Omega}_+(t) \cap \bar{\Omega}_-(t)$ . The motions of the fluids are driven by the constant gravitational field along  $e_3$  (the  $x_3$ -direction),  $G = (0, 0, -g)$  with  $g > 0$  and the Lorentz force induced by the magnetic fields. The motion of the fluids is described by their velocity, pressure and magnetic field functions, which are given for each  $t \geq 0$  by, respectively,

$$(u_{\pm}, \bar{p}_{\pm}, h_{\pm})(t, \cdot) : \Omega_{\pm}(t) \rightarrow (\mathbb{R}^3, \mathbb{R}^+, \mathbb{R}^3).$$

We assume that at a given time  $t \geq 0$ , these functions have well-defined trace onto  $\Sigma(t)$ .

The fluids under consideration are incompressible, viscous and of zero resistivity. Hence for  $t > 0$  and  $x = (x_1, x_2, x_3) \in \Omega_{\pm}(t)$ , the fluids satisfy the following magnetohydrodynamic equations:

$$\begin{cases} \partial_t(\varrho_{\pm} u_{\pm}) + \operatorname{div}(\varrho_{\pm} u_{\pm} \otimes u_{\pm}) + \operatorname{div} S_{\pm} = -g \varrho_{\pm} e_3, \\ \operatorname{div} u_{\pm} = 0, \\ \partial_t h_{\pm} + \operatorname{div}(u_{\pm} \otimes h_{\pm}) - \operatorname{div}(h_{\pm} \otimes u_{\pm}) = 0, \\ \operatorname{div} h_{\pm} = 0, \end{cases} \quad (1.1)$$

where we have defined the stress tensor consisting of both fluid and magnetic parts by

$$S_{\pm} = -\mu_{\pm}(\nabla u_{\pm} + \nabla u_{\pm}^T) + \bar{p}_{\pm} I + \frac{|h_{\pm}|^2}{2} I - h_{\pm} \otimes h_{\pm}.$$

Hereafter the superscribe  $T$  means the transposition and  $I$  is the  $3 \times 3$  identity matrix. The positive constants  $\varrho_{\pm}$  denote the densities of the respective fluids.

For two viscous fluids meeting at a free boundary, the standard assumptions are that the velocity is continuous across the interface and that the jump in the normal stress is proportional to the mean curvature of the surface multiplied by the normal vector to the surface (cf. [2, 19]). This requires us to enforce the jump conditions

$$[u]|_{\Sigma(t)} = 0, \quad (1.2)$$

$$[S\nu]|_{\Sigma(t)} = \kappa H\nu, \quad (1.3)$$

where we have written the normal vector to  $\Sigma(t)$  as  $\nu$ ,  $f|_{\Sigma(t)}$  for the trace of a quantity  $f$  on  $\Sigma(t)$ , and denoted the interfacial jump by

$$[f]|_{\Sigma(t)} := f_+|_{\Sigma(t)} - f_-|_{\Sigma(t)}.$$

Here we take  $H$  to be twice the mean curvature of the surface  $\Sigma(t)$  and the surface tension to be a constant  $\kappa \geq 0$ . We will also enforce the condition that the fluid velocity vanishes at the fixed boundaries; we implement this via the boundary conditions

$$u_+(t, x', -1) = 0, \quad u_-(t, x', 1) = 0 \quad \text{for all } t \geq 0, \quad x' = (x_1, x_2) \in \mathbb{R}^2.$$

Since the fluids are of zero resistivity, the magnetic equations (1.1)<sub>3</sub> are a free transport system along the flow, and hence the Dirichlet boundary condition on the velocity at the fixed boundary prevents the necessity of prescribing boundary condition on the magnetic field. On the other hand, due to the incompressibility (1.1)<sub>4</sub> and also from the physical point of view, we assume that the normal component of the magnetic field is continuous across the free interface (cf. [2, 15])

$$[h \cdot \nu]|_{\Sigma(t)} = 0. \tag{1.4}$$

In fact, we will show in the next subsection that the incompressibility of  $h_{\pm}$  and the jump condition (1.4) are satisfied if they hold initially. Therefore, the conditions (1.1)<sub>4</sub>, (1.4) are transformed to the compatibility conditions assumed on the initial magnetic field.

The motion of the free interface is coupled to the evolution equations for the fluids (1.1) by requiring that the surface be advected with the fluids. More precisely, if  $V(t, x) \in \mathbb{R}^3$  denotes the normal velocity of the boundary at  $x \in \Sigma(t)$ , then

$$V(t, x) = (u(t, x) \cdot \nu(t, x))\nu(t, x),$$

where  $u(t, x)$  is the common trace of  $u_{\pm}(t, x)$  onto  $\Sigma(t)$  and these traces agree because of the jump condition (1.2), which also implies that there is no possibility of the fluids slipping past each other along  $\Sigma(t)$ .

To complete the statement of the problem, we must specify initial conditions. We give the initial interface  $\Sigma(0) = \Sigma_0$  (e.g.,  $\Sigma_0 = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$ ), which yields the open sets  $\Omega_{\pm}(0)$  on which we specify the initial data for the velocity and magnetic field

$$(u_{\pm}, h_{\pm})(0, \cdot) : \Omega_{\pm}(0) \rightarrow (\mathbb{R}^3, \mathbb{R}^3).$$

Thus the initial datum of the pressure  $\bar{p}_0$  can be defined by  $\varrho_{\pm}$ ,  $\Sigma_0$ ,  $u_{\pm}(0, \cdot)$  and  $h_{\pm}(0, \cdot)$ .

To simplify the equations we introduce the indicator function  $\chi$  and denote

$$\begin{aligned} \varrho &= \varrho_+ \chi_{\Omega_+} + \varrho_- \chi_{\Omega_-}, & u &= u_+ \chi_{\Omega_+} + u_- \chi_{\Omega_-}, \\ h &= h_+ \chi_{\Omega_+} + h_- \chi_{\Omega_-}, & \bar{p} &= \bar{p}_+ \chi_{\Omega_+} + \bar{p}_- \chi_{\Omega_-}, \end{aligned}$$

and also define the modified pressure by

$$p = \bar{p} + \frac{|h|^2}{2} + g\varrho x_3.$$

Hence, the equations (1.1) can be rewritten as

$$\begin{cases} \varrho \partial_t u + \varrho u \cdot \nabla u + \nabla p = h \cdot \nabla h + \mu \Delta u, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u = 0, \\ \operatorname{div} u = \operatorname{div} h = 0 \end{cases}$$

in  $\Omega \setminus \Sigma(t)$  for each  $t > 0$ , and the jump condition (1.3) becomes, setting  $[\varrho] = \varrho_+ - \varrho_-$ ,

$$[(pI - \mu(\nabla u + \nabla u^T))\nu]|_{\Sigma(t)} = g[\varrho]x_3\nu + h \cdot \nu[h]|_{\Sigma(t)} + \kappa H\nu.$$

## 1.2. Formulation in Lagrangian coordinates

The movement of the free interface  $\Sigma(t)$  and the subsequent change of the domains  $\Omega_{\pm}(t)$  in Eulerian coordinates will result in severe mathematical difficulties. To circumvent such difficulties, we switch our analysis to Lagrangian coordinates, so that the interface and the domains stay fixed in time. To this end, we define the fixed Lagrangian domains  $\Omega_+ = \mathbb{R}^2 \times (0, 1)$  and  $\Omega_- = \mathbb{R}^2 \times (-1, 0)$ , and assume that there exist invertible mappings

$$\eta_{\pm}^0 : \Omega_{\pm} \rightarrow \Omega_{\pm}(0),$$

such that  $\Sigma_0 = \eta_+^0(\{x_3 = 0\})$ ,  $\{x_3 = 1\} = \eta_+^0(\{x_3 = 1\})$  and  $\{x_3 = -1\} = \eta_-^0(\{x_3 = -1\})$ . The first condition means that  $\Sigma_0$  is parameterized by the mapping  $\eta_+^0$  restricted to  $\{x_3 = 0\}$ , while the latter two conditions mean that  $\eta_{\pm}^0$  map the fixed upper and lower boundaries into themselves. Define the flow maps  $\eta_{\pm}$  as the solution to

$$\begin{cases} \partial_t \eta_{\pm}(t, x) = u_{\pm}(t, \eta_{\pm}(t, x)) \\ \eta_{\pm}(0, x) = \eta_{\pm}^0(x). \end{cases} \quad (1.5)$$

We denote the Eulerian coordinates by  $(t, y)$  with  $y = \eta(t, x)$ , whereas the fixed  $(t, x) \in \mathbb{R}^+ \times \Omega$  stand for the Lagrangian coordinates. In order to switch back and forth from Lagrangian to Eulerian coordinates, we assume that  $\eta_{\pm}(t, \cdot)$  are invertible and  $\Omega_{\pm}(t) = \eta_{\pm}(t, \Omega_{\pm})$ , and since  $u_{\pm}$  and  $\eta_{\pm}^0$  are all continuous across  $\{x_3 = 0\}$ , we have  $\Sigma(t) = \eta_{\pm}(t, \{x_3 = 0\})$ . In other words, the Eulerian domains of upper and lower fluids are the image of  $\Omega_{\pm}$  under the mappings  $\eta_{\pm}$ , and the free interface is the image of  $\{x_3 = 0\}$  under the mappings  $\eta_{\pm}(t, \cdot)$ .

Setting  $\eta = \chi_+ \eta_+ + \chi_- \eta_-$ , we define the Lagrangian unknowns by

$$(v, q, b)(t, x) = (u, p, h)(t, \eta(t, x)), \quad (t, x) \in \mathbb{R}^+ \times \Omega.$$

Defining the matrix  $A := (A_{ij})_{3 \times 3}$  via  $A^T = (D\eta)^{-1} := (\partial_j \eta_i)_{3 \times 3}^{-1}$ , and the identity matrix  $I = (I_{ij})_{3 \times 3}$ , thus in Lagrangian coordinates the evolution equations for  $\eta, v, q, b$  read as (writing  $\partial_j = \partial/\partial x_j$ )

$$\begin{cases} \partial_t \eta_i = v_i \\ \rho \partial_t v_i + A_{jk} \partial_k T_{ij} = b_j A_{jk} \partial_k b_i, \\ A_{jk} \partial_k v_j = 0, \\ \partial_t b_i = b_j A_{jk} \partial_k v_i, \\ A_{jk} \partial_k b_j = 0, \end{cases} \quad (1.6)$$

where the stress tensor of fluid part in Lagrangian coordinates,  $T(v, q)$ , is given by

$$T_{ij} = q I_{ij} - \mu (A_{jk} \partial_k v_i + A_{ik} \partial_k v_j).$$

Here we have written  $I_{ij}$  for the  $ij$ -component of the identity matrix  $I$  and used the Einstein convention of summing over repeated indices.

To write the jump conditions, for a quantity  $f = f_{\pm}$ , we define the interfacial jump by

$$[[f]] := f_+|_{x_3=0} - f_-|_{x_3=0}.$$

Then the jump conditions in Lagrangian coordinates are

$$[[v]] = 0, \quad [[b_j n_j]] = 0, \quad [[T_{ij} n_j]] = g[\rho] \eta_3 n_i + [[b_i]] b_j n_j + \kappa H n_i, \quad (1.7)$$

where we have written  $n := (n_1, n_2, n_3) = \nu(\eta)$ , i.e.

$$n = \frac{\partial_1 \eta \times \partial_2 \eta}{|\partial_1 \eta \times \partial_2 \eta|} = \frac{Ae_3}{|Ae_3|} \Big|_{\{x_3=0\}} \quad (1.8)$$

for the normal vector to the surface  $\Sigma(t) = \eta(t, \{x_3 = 0\})$  and  $H$  for twice the mean curvature of  $\Sigma(t)$ . Since  $\Sigma(t)$  is parameterized by  $\eta$ , we may employ the standard formula for the mean curvature of a parameterized surface to write

$$H = \left( \frac{|\partial_1 \eta|^2 \partial_2^2 \eta - 2(\partial_1 \eta \cdot \partial_2 \eta) \partial_1 \partial_2 \eta + |\partial_2 \eta|^2 \partial_1^2 \eta}{|\partial_1 \eta|^2 |\partial_2 \eta|^2 - |\partial_1 \eta \cdot \partial_2 \eta|^2} \right) \cdot n. \quad (1.9)$$

Finally, we require the no-slip boundary conditions

$$v_-(t, x', -1) = 0, \quad v_+(t, x', 1) = 0 \quad \text{for all } t \geq 0, \quad x' \in \mathbb{R}^2. \quad (1.10)$$

### 1.3. Reformulation

In this subsection we reformulate the free boundary problem (1.6), (1.7) and (1.10). Our goal is to eliminate  $b$  by expressing it in terms of  $\eta$ , and this can be achieved in the same manner as in [18]. For the reader's convenience, we give the derivation here.

Applying  $A_{il}$  to (1.6)<sub>4</sub>, we have

$$A_{il} \partial_t b_i = b_j A_{jk} \partial_k v_i A_{il} = b_j A_{jk} \partial_t (\partial_k \eta_i) A_{il} = -b_j A_{jk} \partial_k \eta_i \partial_t A_{il} = -b_i \partial_t A_{il},$$

which implies that  $\partial_t (A_{jl} b_j) = 0$ . Hence,

$$A_{jl} b_j = A_{jl}^0 b_j^0, \quad (1.11)$$

$$b_i = \partial_t \eta_i A_{jl}^0 b_j^0. \quad (1.12)$$

Hereafter, the superscript 0 means the initial value.

With help of (1.12), and the geometric identities  $J = J^0$ ,  $\partial_k (JA_{ik}) = 0$  where  $J = |D\eta|$ , we can apply  $A_{ik} \partial_k$  to (1.12) to deduce that the divergence of  $b$  (i.e. (1.6)<sub>5</sub>) satisfies

$$A_{ik} \partial_k b_i = \frac{J}{J^0} A_{ik} \partial_k (\partial_t \eta_i A_{jl}^0 b_j^0) = \frac{1}{J^0} \partial_k (JA_{ik} \partial_t \eta_i A_{jl}^0 b_j^0) = \frac{1}{J^0} \partial_k (J^0 A_{jk}^0 b_j^0) = A_{jk}^0 \partial_k b_j^0. \quad (1.13)$$

Next, we evaluate the jump  $\llbracket b_j n_j \rrbracket$ . It is easy, recalling (1.8), to verify that  $Ae_3$  is continuous across the free interface. Hence, one has

$$\llbracket b_j n_j \rrbracket = \llbracket \partial_t \eta_j A_{kl}^0 b_k^0 A_{j3} \rrbracket \frac{1}{|Ae_3|} = \llbracket A_{k3}^0 b_k^0 \rrbracket \frac{1}{|Ae_3|} = \llbracket b_j^0 n_j^0 \rrbracket \frac{|A^0 e_3|}{|Ae_3|}. \quad (1.14)$$

Thus, if one assumes the compatibility conditions on the initial data

$$A_{jk}^0 \partial_k b_j^0 = 0, \quad \llbracket b_j^0 n_j^0 \rrbracket = 0, \quad (1.15)$$

then from (1.13) and (1.14) one gets

$$A_{jk} \partial_k b_j = 0, \quad \llbracket b_j n_j \rrbracket = 0. \quad (1.16)$$

Moreover, for simplifying notation, we assume that

$$A_{ml}^0 b_m^0 = \bar{B}_l \quad \text{with } \bar{B} \text{ being a constant vector.} \quad (1.17)$$

We remark that the class of the pairs of the data  $\eta^0$ ,  $b^0$  that satisfy the constraints (1.15) and (1.17) is quite large. For example, we chose  $\eta^0 = \text{Id}$  and  $b^0 = \text{constant vector}$ , then  $\eta^0$  and  $b^0$  satisfy (1.15) and (1.17), consequently, the pair  $(\eta, b)$  which is transported by the flow will satisfy (1.16).

Now, in view of (1.11), (1.12), (1.17), we represent the Lorentz force term by

$$b_j A_{jk} \partial_k b_i = \partial_l \eta_j A_{ml}^0 b_m^0 A_{jk} \partial_k (\partial_r \eta_i A_{sr}^0 b_s^0) = A_{mk}^0 b_m^0 \partial_k (\partial_r \eta_i A_{sr}^0 b_s^0) = \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta_i.$$

Hence the equations (1.6) become a Navier-Stokes system with the force term induced by the flow map  $\eta$ :

$$\begin{cases} \partial_t \eta_i = v_i \\ \rho \partial_t v_i + A_{jk} \partial_k T_{ij} = \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta_i, \\ A_{jk} \partial_k v_j = 0, \end{cases} \quad (1.18)$$

where the magnetic number  $\bar{B}$  can be regarded as a vector parameter. Accordingly, the jump conditions (1.7) become

$$[[v]] = 0, \quad [[T_{ij} n_j]] = g[\rho] \eta_3 n_i + \bar{B}_l \bar{B}_m [[\partial_l \eta_i]] \partial_m \eta_j n_j + \kappa H n_i. \quad (1.19)$$

Note that we implicitly admit that  $\bar{B}_m \partial_m \eta_j n_j$  is continuous across  $\{x_3 = 0\}$ . In fact, this follows from the assumptions (1.15), (1.17).

Finally, we require the boundary conditions (1.10).

#### 1.4. Linearization around the steady state

The system (1.18), (1.19) and (1.10) admits the steady solution with  $v = 0$ ,  $\eta = \text{Id}$ ,  $q = \text{constant}$  with the interface given by  $\eta(\{x_3 = 0\}) = \{x_3 = 0\}$ , and hence,  $n = e_3$  and  $A = I$ . Now we linearize the equations (1.18) around such a steady-state solution, the resulting linearized equations are

$$\begin{cases} \partial_t \eta = v, \\ \rho \partial_t v + \nabla q = \mu \Delta v + \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta, \\ \text{div} v = 0. \end{cases} \quad (1.20)$$

The corresponding linearized jump conditions are

$$[[v]] = 0, \quad [[qI - \mu(\nabla v + \nabla v^T)]] e_3 = (g[\rho] \eta_3 + \kappa \Delta_{x'} \eta_3) e_3 + \bar{B}_3 \bar{B}_l [[\partial_l \eta]], \quad (1.21)$$

while the boundary conditions are

$$v_-(t, x', -1) = 0, \quad v_+(t, x', 1) = 0. \quad (1.22)$$

Now, we briefly review some of the previous results on the nonlinear Rayleigh-Taylor instability without the magnetic field. In 1987, Erban [5] proved the ill-posedness of the equations of motion for a perfect fluid with free boundary. Then, he adapted the approach of [5] to obtain the nonlinear ill-posedness of both R-T and Helmholtz instability problems for two-dimensional incompressible, immiscible, inviscid fluids without surface tension [4]. In 2011, for two-compressible immiscible fluids evolving with a free interface, Guo and Tice made use of flow maps (cf. (1.5)) to transfer the free boundary into a fixed boundary and established a variational framework for the nonlinear instability in [6], in which with the help of the method of Fourier synthesis, they constructed solutions that grow arbitrarily fast in time in Sobolev spaces, leading to the ill-posedness of the perturbed problem in Lagrangian coordinates. This is in some sense a compressible analogue to the ill-posedness of the R-T instability problem for incompressible fluids given in [5]. Unfortunately, the approaches in both [4] and [6] could not be directly applied to

the viscous flow case, since the viscosity can bring some technical difficulties in the study of the nonlinear R-T instability. Hence, Guo and Tice only investigated the stabilized effect of viscosity and surface tension to the linear R-T instability (see [7]) for compressible flows, and the corresponding nonlinear instability still remains open. Recently, Jiang, Jiang and Wang [10] showed the nonlinear instability in some sense for two incompressible immiscible fluids with or without surface tension in Eulerian coordinates.

Unfortunately, the results concerning the nonlinear R-T instability in [4, 6, 10] are still not generalized to the case of MHD flows, because of additional difficulties induced by presence of the magnetic field. For two incompressible immiscible fluids evolving with a free interface (the density is discontinuous across the interface) under presence of a magnetic field, it was first shown by Kruskal and Schwarzschild [11] that a horizontal magnetic field has no effect on the development of the R-T instability for the linearized equations in the whole space in Eulerian coordinates. Recently, for the case of finite slab domain without surface tension, Wang [18] obtained the critical magnetic number (denoted by  $|B|_c$ ) for the linear R-T instability. Namely, he gave an instability criterion to the linearized problem (1.20) in terms of the value of the magnetic number, and pointed out the linear R-T instability in the case  $\bar{B} = (0, 0, B)$  with  $|B| < |B|_c$ . In particular, unlike 2D case, he also showed that the linearized problem is unstable for any initially horizontal magnetic field  $\bar{B} = (B, 0, 0)$ . From [18], we see that the stabilized effect of the magnetic field is more remarkable than that of the surface tension  $\kappa$  which only stabilizes the frequency interval  $(0, \sqrt{g[\varrho]/\kappa})$  for any  $\kappa > 0$ , also see [7]. Adopting and modifying the approaches in both [6] and [18], Duan, Jiang and Jiang recently showed the ill-posedness of the linearized system (1.20) with  $\mu = 0$  and  $\bar{B} = (B, 0, 0)$  in the sense of Hadamard [3], and moreover, the ill-posedness of the corresponding nonlinear problem in some sense. Finally, we mention that Hwang [8] derived the nonlinear instability around different steady states for both incompressible and compressible inviscid MHD flows with continuous density, thus extending the results in [9] to the case of MHD flows with continuous density.

In this paper, we will study the R-T instability for the problem (1.18), (1.19) and (1.10). We will prove that the corresponding linearized system (1.20)–(1.22) with any  $\bar{B} = (B, 0, 0)$  is globally unstable. Moreover, the original nonlinear problem (1.18) with or without surface tension is globally unstable in some sense. For this purpose, we assume that  $\kappa \geq 0$  and that the upper fluid is heavier than the lower fluid, i.e.,

$$\varrho_+ > \varrho_- \Leftrightarrow [\varrho] > 0.$$

In addition, we will compute that the so-called critical number in [18] indeed is equal  $\sqrt{g[\varrho]}/2$  (see Remark 3.1 for the details), and analyze the effect of viscosity and surface tension on the linear instability (see Remark 3.2). The results of the current paper extend the ones in [8, 18].

The rest of this paper is organized as follows. In Section 2 we state our results concerning the linearized system (1.20) and nonlinear system (1.18), i.e. Theorems 2.1 and 2.2, respectively. In Section 3 we construct the growing solutions to the linearized equations, while in Section 4 we prove the uniqueness of the linearized problem and Theorem 2.1. In Section 5, we prove the global instability of order  $k$  to the corresponding nonlinear problem, i.e., Theorem 2.2.

## 2. Main results

Before stating the main results, we introduce the notation that will be used throughout the paper. For a function  $f \in L^2(\Omega)$ , we define the horizontal Fourier transform via

$$\hat{f}(\xi, x_3) = \int_{\mathbb{R}^2} f(x', x_3) e^{-ix' \cdot \xi} dx',$$

where  $x', \xi \in \mathbb{R}^2$  and “ $\cdot$ ” denotes scalar product. By the Fubini and Parseval theorems, we have

$$\int_{\Omega} |f(x)|^2 dx = \frac{1}{4\pi^2} \int_{\Omega} |\hat{f}(\xi, x_3)|^2 d\xi dx_3.$$

We now define a function space suitable for our analysis of two disjoint fluids. For a function  $f$  defined on  $\Omega$ , we write  $f_+$  for the restriction to  $\Omega_+ = \mathbb{R}^2 \times (0, 1)$  and  $f_-$  for the restriction to  $\Omega_- = \mathbb{R}^2 \times (-1, 0)$ . For  $s \in \mathbb{R}$ , define the piecewise Sobolev space of order  $s$  by

$$\bar{H}^s(\Omega) = \{f \mid f_+ \in H^s(\Omega_+), f_- \in H^s(\Omega_-)\} \quad (2.1)$$

endowed with the norm  $\|f\|_{\bar{H}^s}^2 = \|f\|_{H^s(\Omega_+)}^2 + \|f\|_{H^s(\Omega_-)}^2$ . For  $k \in \mathbb{N}$  we can take the  $\bar{H}^s$ -norm to be

$$\begin{aligned} \|f\|_{\bar{H}^k}^2 &:= \sum_{j=0}^k \int_{\mathbb{R}^2 \times I_{\pm}} (1 + |\xi|^2)^{k-j} \left| \partial_{x_3}^j \hat{f}_{\pm}(\xi, x_3) \right|^2 d\xi dx_3 \\ &= \sum_{j=0}^k \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} \left\| \partial_{x_3}^j \hat{f}_{\pm}(\xi, \cdot) \right\|_{L^2(I_{\pm})}^2 d\xi \end{aligned}$$

for  $I_- = (-1, 0)$  and  $I_+ = (0, 1)$ . The main difference between the piecewise Sobolev space  $\bar{H}^s(\Omega)$  and the usual Sobolev space  $H^s(\Omega)$  is that we do not require functions in the piecewise Sobolev space to have weak derivatives across the interface  $\{x_3 = 0\}$ .

Now, we are in a position to state our first result, i.e. the result of global instability for the linearized problem (1.20).

**Theorem 2.1.** *Assume the constant vector  $\bar{B} = (B, 0, 0)$ , then the linearized problem (1.20) with the corresponding jump and boundary conditions (1.21), (1.22) is globally unstable in  $\bar{H}^k(\Omega)$  for every  $k$ . More precisely, there exists a constant  $C_1 > 0$ , and for any  $k, j \in \mathbb{N}$  with  $j \geq k$  and for any  $\alpha > 0$ , there exist a constant  $C_{j,k}$  depending on  $j$  and  $k$ , and a sequence of solutions  $\{(\eta_n, v_n, q_n)\}_{n=1}^{\infty}$  to (1.20) satisfying the corresponding jump and boundary conditions (1.21), (1.22), such that*

$$\|\eta_n(0)\|_{\bar{H}^j} + \|v_n(0)\|_{\bar{H}^j} + \|q_n(0)\|_{\bar{H}^j} \leq \frac{1}{n}, \quad (2.2)$$

but

$$\|v_n(t)\|_{\bar{H}^k} \geq \alpha \text{ for all } t \geq t_n := C_{j,k} + C_1 \ln(\alpha n). \quad (2.3)$$

Moreover, there exists a positive constant  $\lambda_0$  such that

$$\|\eta_n(t)\|_{\bar{H}^k} \geq \lambda_0 \|v_n(t)\|_{\bar{H}^k} \text{ and } \|v_n(t)\|_{\bar{H}^k} \rightarrow \infty \text{ as } t \rightarrow \infty. \quad (2.4)$$

Theorem 2.1 shows globally discontinuous dependence of solutions upon initial data. The proof of Theorem 2.1 is motivated by [7] under nontrivial modifications, and its basic idea is the following. First, noticing that the coefficients of the linearized equations depend only on the vertical variable  $x_3 \in (-1, 1)$ , we seek “normal mode” solutions by taking the horizontal Fourier transform of the equations and assuming that the solutions grow exponentially in time with the factor  $e^{\lambda(|\xi_1|, |\xi_2|)t}$ , where  $\xi \in \mathbb{R}^2$  is the horizontal spatial frequency and  $\lambda(|\xi_1|, |\xi_2|) > 0$ . This reduces the equations to a system of ordinary differential equations (ODE) with  $\lambda(|\xi_1|, |\xi_2|) > 0$  for each  $\xi$ . Thus, solving the ODE system by the modified variational method, we show that  $\lambda(|\xi_1|, |\xi_2|) > 0$  is a continuous function on some symmetric open domain  $\mathbb{A} \subset \{\xi \in \mathbb{R}^2 \mid |\xi| \in (0, |\xi|_c)\}$  (see Proposition 3.7), the normal modes with spatial frequency grow in time, providing a mechanism for the global R-T instability, where  $|\xi|_c = \sqrt{g[\varrho]}/\kappa$  if  $\kappa > 0$ , otherwise  $|\xi|_c = \infty$ .



In particular, we can restrict  $\xi$  in some symmetric closed sector domain  $\mathbb{D}$ , such that  $\lambda(|\xi|)$  has a uniform lower bound for any  $\xi \in \mathbb{D}$  (see Remark 3.3). Then, we form a Fourier synthesis of the normal mode solutions constructed for each spatial frequency  $\xi$  to give solutions of the linearized equations that grow in time, when measured in  $\bar{H}^k(\Omega)$  for any  $k \geq 0$ . Finally, we exploit the property of the trace theorem to show a uniqueness result of the linearized problem (i.e. Theorem 4.1). Our results show that in spite of the uniqueness, the linearized problem is still globally unstable in  $\bar{H}^k(\Omega)$  for any  $k$ .

With the linear global instability established, we can show the global instability of the corresponding nonlinear problem in some sense (i.e. Theorem 2.2). Recalling that the steady state solution to (1.20) is given by  $\eta = 0$ ,  $v = 0$ ,  $q = \text{constant}$ , we now rewrite the nonlinear equations (1.18) in the form of perturbation around the steady state. Let

$$\eta = \text{Id} + \tilde{\eta} \text{ is invertible, } q = \text{constant} + \tilde{q}, \quad v = 0 + v, \quad A = I - G,$$

where

$$G^T = I - (I + D\tilde{\eta})^{-1}.$$

Then, the evolution equations (1.18) can be rewritten for  $\tilde{\eta}$ ,  $v$ ,  $\tilde{q}$  as

$$\begin{cases} \partial_t \tilde{\eta} = v, \\ \varrho \partial_t v_i + (I_{jk} - G_{jk}) \partial_k \tilde{T}_{ij} = \bar{B}_l \bar{B}_m \partial_{lm}^2 \tilde{\eta}_i, \quad i = 1, 2, 3, \\ \text{div} v - \text{tr}(GDv) = 0, \end{cases} \quad (2.5)$$

where  $\text{tr}(\cdot)$  denotes the matrix trace and

$$\tilde{T}_{ij} = \tilde{q} I_{ij} - \mu((I_{jk} - G_{jk}) \partial_k v_i + (I_{ik} - G_{ik}) \partial_k v_j).$$

The jump conditions across the interface are

$$[[v]] = 0, \quad [[\tilde{T}_{ij} n_j]] = g[\varrho] \tilde{\eta}_3 n_i + \bar{B}_l \bar{B}_m [[\partial_l \tilde{\eta}_i]] (I_{ij} + \partial_m \tilde{\eta}_j) n_j + \kappa H n_i, \quad (2.6)$$

where  $n$  and  $H$  are respectively given by (1.8) and (1.9) with  $\eta = \text{Id} + \tilde{\eta}$ . Finally, we require the boundary conditions

$$v_-(t, x', -1) = 0, \quad v_+(t, x', 1) = 0. \quad (2.7)$$

We collectively refer to the evolution, jump, and boundary equations (2.5)–(2.7) as “the perturbed problem”. To shorten notation, for  $k \geq 0$  we define

$$\|(\tilde{\eta}, v, \tilde{q})(t)\|_{\bar{H}^k} = \|\tilde{\eta}(t)\|_{\bar{H}^k} + \|v(t)\|_{\bar{H}^k} + \|\tilde{q}(t)\|_{\bar{H}^k}.$$

**Definition 2.1.** *We say that the perturbed problem has global stability of order  $k$  for some  $k \geq 3$  if there exist  $\delta$ ,  $C_2 > 0$  and a function  $F : [0, \delta) \rightarrow \mathbb{R}^+$  satisfying  $F(z) \leq C_2 z$  for  $z \in [0, \delta)$ , so that the followings hold. For any  $T > 0$ ,  $\tilde{\eta}_0$ , and  $v_0$  satisfying*

$$\|(\tilde{\eta}_0, v_0)\|_{\bar{H}^k} < \delta,$$

*there exist  $\tilde{\eta}, v, \tilde{q} \in L^\infty(0, T; \bar{H}^3(\Omega))$ , so that*

- (1)  $(\tilde{\eta}, v)(0) = (\tilde{\eta}_0, v_0)$ ;
- (2)  $\tilde{\eta}, v, \tilde{q}$  solve the perturbed problem on  $(0, T) \times \Omega$ ;
- (3) it holds that

$$\sup_{0 \leq t \leq T} \|(\tilde{\eta}, v, \tilde{q})(t)\|_{\bar{H}^3} \leq F(\|(\tilde{\eta}_0, v_0)\|_{\bar{H}^k});$$

- (4)  $\eta_\pm \in C^2(\bar{\Omega}_\pm)$ , respectively, when  $\kappa > 0$ .

**Remark 2.1.** In the above definition, in view of (1.8) and (1.9), we have admitted that

$$n = \frac{\partial_1 \eta_{\pm} \times \partial_2 \eta_{\pm}}{|\partial_1 \eta_{\pm} \times \partial_2 \eta_{\pm}|} \Big|_{\{x_3=0\}}$$

and

$$H = \left( \frac{|\partial_1 \eta_{\pm}|^2 \partial_2^2 \eta_{\pm} - 2(\partial_1 \eta_{\pm} \cdot \partial_2 \eta_{\pm}) \partial_1 \partial_2 \eta_{\pm} + |\partial_2 \eta_{\pm}|^2 \partial_1^2 \eta_{\pm}}{|\partial_1 \eta_{\pm}|^2 |\partial_2 \eta_{\pm}|^2 - |\partial_1 \eta_{\pm} \cdot \partial_2 \eta_{\pm}|^2} \right) \cdot n \Big|_{\{x_3=0\}}$$

with  $\eta_{\pm} = \text{Id} + \tilde{\eta}_{\pm}$ . For convenience in the subsequent proof of Theorem 2.2, we will still use  $\eta$  to denote the both  $\eta_-$  and  $\eta_+$  at the interface  $\{x_3 = 0\}$ . Similarly,  $\tilde{\eta}$  in (2.6) also includes the both cases of  $\tilde{\eta}_-$  and  $\tilde{\eta}_+$  at  $\{x_3 = 0\}$  except for the notation  $[\partial_i \tilde{\eta}_i]$ . We point out that Theorem 2.2 below still holds if  $\tilde{\eta}$  only denotes  $\tilde{\eta}_+$  (or  $\tilde{\eta}_-$ ) at  $\{x_3 = 0\}$ .

We can show that the property of global stability of order  $k$  cannot hold for any  $k \geq 3$ , i.e., the following Theorem 2.2, which will be proved in Section 5.

**Theorem 2.2.** *Assume the constant vector  $\bar{B} = (B, 0, 0)$ , the perturbed problem does not have property of global stability of order  $k$  for any  $k \geq 3$ .*

The basic idea in the proof of Theorem 2.2 is to show, by utilizing the Lipschitz structure of  $F$ , that the global stability of order  $k$  would give rise to certain estimates of solutions to the linearized equations (1.18) that cannot hold in general because of Theorem 2.1. We will adapt and modify the arguments in [6] to prove Theorem 2.2. Compared with the perturbed problem in [6, Theorem 5.2], the main difficulty lies in the convergence of the jump conditions of the rescaled pressure function sequence, because we could not obtain the strong convergence of the rescaled pressure as in [6]. To circumvent such difficulty, similarly to [10], we apply the techniques of integrating by parts to avoid using the strong convergence of the rescaled pressure. Also, this idea is used in the proof of the uniqueness of solutions to the linearized equations (1.20) in Section 4.

**Remark 2.2.** Theorems 2.1 and 2.2 show that a horizontal magnetic field can not prevent the linear and nonlinear R-T global instability in the sense described in Theorems 2.1 and 2.2. From the proof we easily see that Theorems 2.1 and 2.2 still hold with  $\bar{B} = (0, B, 0)$  in place of  $\bar{B} = (B, 0, 0)$ . In addition, based on the results in [18], we can show that Theorems 2.1 and 2.2 still remain valid with  $\bar{B} = (0, 0, B)$  in place of  $\bar{B} = (B, 0, 0)$ , if one of the following two conditions holds:

- (1)  $\kappa = 0$  and  $|B| < g[\varrho]/2$ ;
- (2)  $\kappa > 0$  and  $|B|$  sufficiently small.

### 3. Construction of a growing solution to linearized equations

#### 3.1. Growing mode ansatz and the horizontal Fourier transform

We wish to construct a solution to the linearized equations (1.20) that has a growing  $H^k$  norm for any  $k$ . We will construct such solutions via Fourier synthesis by first constructing a growing mode for fixed spatial frequency.

To begin, we assume a growing mode ansatz (cf. [2]), i.e.,

$$v(t, x) = w(x)e^{\lambda t}, \quad q(t, x) = \varpi(x)e^{\lambda t}, \quad \eta(t, x) = \varsigma(x)e^{\lambda t}$$

for some  $\lambda > 0$ , where  $w = (w_1, w_2, w_3)$ . Substituting this ansatz into (1.20), eliminating  $\varsigma$  by using the first equation, we arrive at the time-independent system for  $w$  and  $\varpi$ :

$$\begin{cases} \lambda \varrho w + \nabla \varpi = \mu \Delta w + \lambda^{-1} \bar{B}_l \bar{B}_m \partial_{lm}^2 w, \\ \operatorname{div} w = 0, \end{cases} \quad (3.1)$$

with the corresponding jump conditions

$$[[w]] = 0, \quad [[\varpi I - \mu(Dw + Dw^T)]]e_3 = \lambda^{-1}(g[\varrho]w_3 + \kappa \Delta_{x'} w_3)e_3 + \lambda^{-1} \bar{B}_3 \bar{B}_l [[\partial_l w]],$$

and the boundary conditions

$$w(t, x', -1) = 0, \quad w(t, x', 1) = 0.$$

We take the horizontal Fourier transform of  $w_1, w_2, w_3$  in (3.1), which we denote by either  $\hat{\cdot}$  or  $\mathcal{F}$ , and fix a spatial frequency  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ . Define the new unknowns  $\varphi(x_3) = i\hat{w}_1(\xi, x_3)$ ,  $\theta(x_3) = i\hat{w}_2(\xi, x_3)$ ,  $\psi(x_3) = \hat{w}_3(\xi, x_3)$  and  $\pi(x_3) = \hat{\varpi}_3(\xi, x_3)$ , so that

$$\mathcal{F}(\operatorname{div} w) = \xi_1 \varphi + \xi_2 \theta + \psi',$$

where  $' = d/dx_3$ . Since we consider the case  $\bar{B} = (\bar{B}_1, \bar{B}_2, \bar{B}_3) \equiv (B, 0, 0)$ , then for  $\varphi, \theta, \psi$  and  $\lambda = \lambda(\xi)$  we arrive at the following system of ODEs.

$$\begin{cases} \lambda^2 \varrho \varphi - \lambda \xi_1 \pi + \lambda \mu(|\xi|^2 \varphi - \varphi'') + B^2 \xi_1^2 \varphi = 0, \\ \lambda^2 \varrho \theta - \lambda \xi_2 \pi + \lambda \mu(|\xi|^2 \theta - \theta'') + B^2 \xi_1^2 \theta = 0, \\ \lambda^2 \varrho \psi + \lambda \pi' + \lambda \mu(|\xi|^2 \psi - \psi'') + B^2 \xi_1^2 \psi = 0, \\ \xi_1 \varphi + \xi_2 \theta + \psi' = 0, \end{cases} \quad (3.2)$$

along with the jump conditions

$$\begin{cases} [[\varphi]] = [[\theta]] = [[\psi]] = 0, \\ [[\lambda \mu(\xi_1 \psi - \varphi')]] = [[\lambda \mu(\xi_2 \psi - \theta')]] = 0, \\ [[-2\lambda \mu \psi' + \lambda \pi]] = (g[\varrho] - k|\xi|^2)\psi, \end{cases} \quad (3.3)$$

and the boundary conditions

$$\varphi(-1) = \varphi(1) = \theta(-1) = \theta(1) = \psi(-1) = \psi(1) = 0. \quad (3.4)$$

Eliminating  $\pi$  from the third equation in (3.2), we obtain the following ODE for  $\psi$

$$-\lambda^2 \varrho(|\xi|^2 \psi - \psi'') = \lambda \mu(|\xi|^4 \psi - 2|\xi|^2 \psi'' + \psi''''') + B^2 \xi_1^2(|\xi|^2 \psi - \psi'') \quad (3.5)$$

along with the jump conditions

$$[[\psi]] = [[\psi']] = 0, \quad (3.6)$$

$$[[\lambda \mu(|\xi|^2 \psi + \psi'')]] = 0, \quad (3.7)$$

$$[[\lambda \mu(\psi''' - 3|\xi|^2 \psi')]] = [[\lambda^2 \varrho \psi']] + (g[\varrho] - k|\xi|^2)|\xi|^2 \psi, \quad (3.8)$$

and the boundary conditions

$$\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0. \quad (3.9)$$

Before constructing a growing solution to ODE of (3.5)–(3.9) in next subsection, we should introduce so-called admissible set for a growing solution.

First we define the maximal frequency by

$$|\xi|_c := \begin{cases} \sqrt{g[\varrho]/\kappa}, & \text{if } \kappa > 0, \\ +\infty, & \text{if } \kappa = 0, \end{cases} \quad (3.10)$$

and the generalized critical magnetic number (depending on the frequency) by

$$B_c(|\xi|) := \sqrt{\sup_{\substack{\psi \in H_0^1(-1,1) \\ \psi \not\equiv 0}} \frac{(g[\varrho] - \kappa|\xi|^2)\psi^2(0)}{\int_{-1}^1 |\psi'|^2 dy}} \quad \text{for } |\xi| < |\xi|_c, \quad (3.11)$$

and the generalized critical frequency function by

$$S(|\xi_1|, |\xi_2|) = \sqrt{\sup_{\substack{\psi \in H_0^1(-1,1) \\ \psi \not\equiv 0}} \frac{(g[\varrho] - \kappa|\xi|^2)\psi^2(0) - f^2(|\xi_1|, |\xi_2|) \int_{-1}^1 |\psi'|^2 dy}{f^2(|\xi_1|, |\xi_2|) \int_{-1}^1 |\psi|^2 dy}} \quad (3.12)$$

for  $|\xi| < |\xi|_c$  and  $0 < f(\xi) < B_c(|\xi|)$ , where  $f(|\xi_1|, |\xi_2|) = |\xi_1 B|/|\xi|$  and  $S(|\xi_1|, |\xi_2|) := \infty$  when  $\xi_1 B = 0$ .

By [18, Lemma 3.2], we see that the supremums in both (3.11) and (3.12) are achieved for each  $\xi$  and  $B$  with  $\xi_1 B \neq 0$ . So, (3.11) and (3.12) make sense. Moreover  $S(|\xi_1|, |\xi_2|) \rightarrow \infty$  as  $f(|\xi_1|, |\xi_2|) \rightarrow 0$ .

Now, we define an admissible set for a growing solution

$$\mathbb{A} = \{\xi \in \mathbb{R}^2 \mid |\xi| < |\xi|_c, 0 < f(|\xi_1|, |\xi_2|) < B_c(|\xi|), 0 < |\xi| < S(|\xi_1|, |\xi_2|)\}. \quad (3.13)$$

By construction, the admissible set possesses the following properties, which will be useful in Subsections 3.2–3.4.

**Proposition 3.1.** *Assume  $\mathbb{A}$  is defined by (3.13), then*

- (1) *the set  $\mathbb{A}$  is a open set in  $\mathbb{R}^2$ ;*
- (2) *the set  $\mathbb{A}$  is symmetric on  $x$ -axis and  $y$ -axis in  $\mathbb{R}^2$ , respectively;*
- (3) *there exist three positive constants  $d_1 < d_2$  and  $d_3$ , such that*

$$\mathbb{D} := \{\xi \in \mathbb{R}^2 \mid d_1 \leq |\xi| \leq d_2, |\xi_1| \leq d_3\} \subset \mathbb{A}. \quad (3.14)$$

PROOF. The proof is very elementary and we omit it here.

In the next subsection, we will show that for any fixed  $B$  and any  $\xi \in \mathbb{A}$ , there is a nontrivial solution  $\psi$  with  $\lambda > 0$  to the problem (3.5)–(3.9). The next proposition shows that  $B_c(|\xi|)$  can be in fact represented by  $g$ ,  $[\varrho]$ ,  $\kappa$  and  $|\xi|$ .

**Proposition 3.2.** *Let  $B_c(|\xi|)$  be defined by (3.11). Then,  $B_c(|\xi|) = \sqrt{(g[\varrho] - \kappa|\xi|^2)/2}$ .*

PROOF. By virtue of (3.11), it suffices to prove that

$$\sup_{\psi \in H_0^1(-1,1)} \frac{\psi^2(0)}{\int_{-1}^1 |\psi'|^2 dy} = \frac{1}{2}. \quad (3.15)$$

For any  $\psi \in H_0^1(-1, 1)$ , we have  $\psi(0) = \int_{-1}^0 \psi' dy = \int_1^0 \psi' dy$ , which gives

$$|\psi(0)| \leq \frac{1}{2} \int_{-1}^1 |\psi'| dy.$$

Hence, by the Hölder inequality, we see that

$$\frac{\psi^2(0)}{\int_{-1}^1 |\psi'|^2 dy} \leq \frac{1}{4} \frac{\left( \int_{-1}^1 |\psi'| dy \right)^2}{\int_{-1}^1 |\psi'|^2 dy} \leq \frac{1}{2}. \quad (3.16)$$

On the other hand, it is easy to check that the function

$$\psi_s = \begin{cases} 1+x, & x \in (-1, 0], \\ 1-x, & x \in (0, 1) \end{cases}$$

satisfies  $\psi_s \in H_0^1(-1, 1)$ . Furthermore, a simple computation results in

$$\frac{\psi_s^2(0)}{\int_{-1}^1 |\psi_s'|^2 dy} = \frac{1}{2},$$

which combined with (3.16) implies (3.15).  $\square$

**Remark 3.1.** We mention that the definition of critical number and critical frequency (both are independent of frequency  $\xi$ ) was introduced by Wang [18], where he defined the critical number by

$$|B|_c := \sqrt{\sup_{\psi \in H_0^1(-1, 1)} \frac{g[\varrho] \psi^2(0)}{\int_{-1}^1 |\psi'|^2 dy}},$$

which is equal to  $\sqrt{g[\varrho]/2}$  by virtue of Proposition 3.2.

### 3.2. Construction of a growing solution to ODE

Throughout this subsection we assume that  $\xi \in \mathbb{A}$ . Similarly to [7, 18], we can apply the variational method to construct a growing solution of (3.5)–(3.9) for given  $\xi \in \mathbb{A}$ . For the reader's convenience, we sketch the procedure of the construction.

First of all, in order to apply the variational method, we remove the linear dependence on  $\lambda$  in (3.5)–(3.8) by defining the modified viscosities  $\bar{\mu} = s\mu$ , where  $s > 0$  is an arbitrary parameter. Thus we obtain a family ( $s > 0$ ) of modified problems

$$-\lambda^2 \rho(|\xi|^2 \psi - \psi'') = s\mu(|\xi|^4 \psi - 2|\xi|^2 \psi'' + \psi'''' ) + B^2 \xi_1^2 (|\xi|^2 \psi - \psi'') \quad (3.17)$$

along with the jump conditions

$$[\![\psi]\!] = [\![\psi']]\!] = 0, \quad (3.18)$$

$$[s\mu(|\xi|^2 \psi + \psi'')] = 0, \quad (3.19)$$

$$[s\mu(\psi''' - 3|\xi|^2 \psi')] = [\![\lambda^2 \varrho \psi']]\!] + (g[\rho] - \kappa|\xi|^2)|\xi|^2 \psi, \quad (3.20)$$

and the boundary conditions

$$\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0. \quad (3.21)$$

Notice that for any fixed  $s > 0$  and  $\xi$ , (3.17)–(3.21) is a standard eigenvalue problem for  $-\lambda^2$ . This allows us to use the variational method to construct solutions. We define the energies by

$$E(\psi) = \frac{1}{2} \int_{-1}^1 s\mu(4|\xi|^2|\psi'|^2 + |\xi|^2\psi + \psi''|^2) + B^2\xi_1^2(|\psi'|^2 + |\xi|^2\psi^2)dx_3 - \frac{1}{2}|\xi|^2(g[\rho] - \kappa|\xi|^2)\psi^2(0), \quad (3.22)$$

and

$$J(\psi) = \frac{1}{2} \int_{-1}^1 \rho(|\xi|^2\psi^2 + |\psi'|^2)dx_3 \quad (3.23)$$

which are well-defined on the space  $H_0^2(-1, 1)$ . We define the admissible set for the energy (3.22)

$$\mathcal{A} = \{\psi \in H_0^2(-1, 1) \mid J(\psi) = 1\}.$$

Thus we can find the smallest  $-\lambda^2$  by minimizing

$$-\lambda^2(|\xi_1|, |\xi_2|) = \alpha(|\xi_1|, |\xi_2|) := \inf_{\psi \in \mathcal{A}} E(\psi). \quad (3.24)$$

In fact, we can show that a minimizer of (3.24) exists and satisfies the Euler-Lagrange equations equivalent to (3.17)–(3.21).

**Proposition 3.3.** *For any fixed  $\xi \in \mathbb{A}$ ,  $E$  achieves its infimum on  $\mathcal{A}$ . In addition, let  $\psi$  be a minimizer and  $-\lambda^2 := E(\psi)$ , then the pair  $(\psi, \lambda^2)$  satisfies (3.17) along with the jump and boundary conditions (3.18)–(3.21). Moreover,  $\psi$  is smooth when restricted to  $(-1, 0)$  or  $(0, 1)$ .*

PROOF. We can follow the same proof procedure as in [18, Proposition 3.1] to obtain Proposition 3.1. Hence, we omit the details of the proof here.  $\square$

In order to emphasize the dependence on  $s \in (0, \infty)$ , we will sometimes write

$$\lambda(|\xi_1|, |\xi_2|) = \lambda(|\xi_1|, |\xi_2|, s)$$

and

$$\alpha(|\xi_1|, |\xi_2|) = \alpha(s).$$

Next, we want to prove that there is a fixed point such that  $\lambda = s$ . To this end, we first give some properties of  $\alpha(s)$  as a function of  $s > 0$  for each fixed  $\xi \in \mathbb{A}$ .

**Proposition 3.4.** *Let  $\alpha(s) : (0, \infty) \rightarrow \mathbb{R}$  be given by (3.24), then the following conclusions hold:*

- (1)  $\alpha(s) \in C_{\text{loc}}^{0,1}(0, \infty)$  is strictly increasing;
- (2) there exist two positive constants  $c_1 := c_1(g, \varrho_{\pm})$  and  $c_2 := c_2(|\xi|, \mu_{\pm})$ , such that

$$\alpha(s) \geq -c_1|\xi| + sc_2;$$

- (3) for each fixed  $\xi \in \mathbb{A}$ , there exist two positive constants  $c_3 := c_3(|B|, |\xi|, g, \varrho_{\pm}, \kappa)$  and  $c_4 := c_4(|B|, |\xi|, g, \varrho_{\pm}, \mu_{\pm}, \kappa)$ , such that

$$\alpha(s) \leq -c_3 + sc_4. \quad (3.25)$$

PROOF. The first two assertions can be shown in the same way as in [18, Lemma 3.5] and [7, Propostition 3.6]. It remains to prove the third assertion to complete the proof.

First, observe that the energy  $E(\psi)$  can be decomposed in the form:

$$E(\psi) = |\xi|^2 E_0(\psi) + s E_1(\psi),$$

where

$$\begin{aligned} E_0(\psi) &= \frac{1}{2} \int_{-1}^1 f^2(|\xi_1|, |\xi_2|) (|\psi'|^2 + |\xi|^2 \psi^2) dx_3 - \frac{1}{2} (g[\varrho] - \kappa |\xi|^2) \psi^2(0), \\ E_1(\psi) &= \frac{1}{2} \int_{-1}^1 \mu (4|\xi|^2 |\psi'|^2 + ||\xi|^2 \psi + \psi''|^2) dx_3, \\ f(|\xi_1|, |\xi_2|) &= \frac{|\xi_1 B|}{|\xi|}. \end{aligned}$$

Then, since  $\xi \in \mathbb{A}$  by virtue of the definitions (3.12) and (3.13), there exists  $\bar{\psi} \in H_0^1$  such that

$$\int_{-1}^1 f^2(|\xi_1|, |\xi_2|) (|\xi|^2 \bar{\psi}^2 + |\bar{\psi}'|^2) dx_3 < (g[\varrho] - \kappa |\xi|^2) \bar{\psi}^2(0). \quad (3.26)$$

On the other hand,  $C_0^\infty(-1, 1)$  is dense in  $H_0^1(-1, 1)$ , then there exists a function sequence  $\bar{\psi}_n \in C_0^\infty(-1, 1)$ , so that

$$\bar{\psi}_n \rightarrow \bar{\psi} \text{ strongly in } H_0^1(-1, 1), \quad (3.27)$$

which, together with the compact embedding theorem, yields

$$\bar{\psi}_n \rightarrow \bar{\psi} \text{ strongly in } C^0(-1, 1). \quad (3.28)$$

Putting (3.26)–(3.28) together, we see that there exist a subsequence  $\bar{\psi}_{n_0} \in H_0^2(-1, 1)$ , such that  $\bar{\psi}_{n_0} \not\equiv 0$  and

$$\int_{-1}^1 f^2(|\xi_1|, |\xi_2|) (|\xi|^2 |\bar{\psi}_{n_0}|^2 + |\bar{\psi}_{n_0}'|^2) dx_3 < (g[\varrho] - \kappa |\xi|^2) \bar{\psi}_{n_0}^2(0), \quad (3.29)$$

which implies

$$E_0(\bar{\psi}_{n_0}) < 0.$$

Thus, we have

$$\begin{aligned} \alpha(s) &= \inf_{\psi \in \mathcal{A}} E(\psi) = \inf_{\substack{\psi \in H_0^2(-1, 1) \\ \psi \not\equiv 0}} \frac{E(\psi)}{J(\psi)} \\ &\leq \frac{E(\bar{\psi}_{n_0})}{J(\bar{\psi}_{n_0})} = |\xi|^2 \frac{E_0(\bar{\psi}_{n_0})}{J(\bar{\psi}_{n_0})} + s \frac{E_1(\bar{\psi}_{n_0})}{J(\bar{\psi}_{n_0})} := -c_3 + s c_4 \end{aligned} \quad (3.30)$$

for two positive constants  $c_3 := c_3(|B|, |\xi|, g, \varrho_\pm, \kappa)$  and  $c_4 := c_4(|B|, |\xi|, g, \varrho_\pm, \mu_\pm, \kappa)$ .  $\square$

Given  $\xi \in \mathbb{A}$ , by virtue of (3.25), there exists a  $s_0 > 0$  depending on the quantities  $|B|, |\xi|, g, \varrho_\pm, \mu_\pm$  and  $\kappa$ , so that for  $s \leq s_0$ ,  $\alpha(s) < 0$ . Therefore, we can define the open set

$$\mathcal{S} = \alpha^{-1}(-\infty, 0) \subset (0, \infty).$$

Note that  $\mathcal{S}$  is non-empty and this allows us to define  $\lambda(s) = \sqrt{-\alpha(s)}$  for  $s \in \mathcal{S}$ . Thus, as a result of Proposition 3.1, we have the following existence result for the modified problem (3.17)–(3.21).

**Proposition 3.5.** *For each  $\xi \in \mathbb{A}$  and each  $s \in \mathcal{S}$ , there is a solution  $\psi = \psi(|\xi_1|, |\xi_2|, x_3)$  with  $\lambda = \lambda(|\xi_1|, |\xi_2|, s) > 0$  to the problem (3.17) with the jump and boundary conditions (3.18)–(3.21). Moreover,  $\psi$  is smooth when restricted to  $(-1, 0)$  or  $(0, 1)$  with  $\psi(|\xi_1|, |\xi_2|, 0) \neq 0$ .*

Finally, we can use Proposition 3.4 to make a fixed-point argument to find a  $s \in \mathcal{S}$  satisfying  $s = \lambda(|\xi_1|, |\xi_2|, s)$  and to construct solutions to the original problem (3.2)–(3.4).

**Proposition 3.6.** *Let  $\xi \in \mathbb{A}$ , then there exists a unique  $s \in \mathcal{S}$ , so that  $\lambda(|\xi_1|, |\xi_2|, s) = \sqrt{-\alpha(s)} > 0$  and  $s = \lambda(|\xi_1|, |\xi_2|, s)$ .*

PROOF. We refer to [18, Lemma 3.7] for a proof.  $\square$

Consequently, in view of Propositions 3.5 and 3.6, we conclude the following existence result on the problem (3.2)–(3.4).

**Theorem 3.1.** *For each  $\xi \in \mathbb{A}$ , there exist  $\psi = \psi(|\xi_1|, |\xi_2|, x_3)$  and  $\lambda(|\xi_1|, |\xi_2|) > 0$  satisfying (3.2)–(3.4). Moreover,  $\psi$  is smooth when restricted to  $(-1, 0)$  or  $(0, 1)$  with  $\psi(|\xi_1|, |\xi_2|, 0) \neq 0$ .*

Next, we show some properties of the solutions established in Theorem 3.1 in terms of  $\lambda(\xi) := \lambda(|\xi_1|, |\xi_2|)$ . The first property is given in the following proposition which shows that  $\lambda$  is a bounded, continuous function of  $\xi \in \mathbb{A}$ . To this end, we shall apply the following Ehrling-Nirenberg-Gagliardo interpolation inequality, the proof of which can be found in [1, Chapter 5].

**Lemma 3.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$  satisfying the cone condition. For each  $\varepsilon_0 > 0$  there is a constant  $K$  depending on  $n, m, p$  and  $\varepsilon_0$ , such that for  $0 < \varepsilon \leq \varepsilon_0$ ,  $0 \leq j \leq m$  and  $u \in W^{m,p}(\Omega)$ ,*

$$\sum_{|\alpha|=j} \int_{\Omega} |D^{\alpha} u(x)|^p dx \leq K \left( \varepsilon \sum_{|\alpha|=m} \int_{\Omega} |D^{\alpha} u(x)|^p dx + \varepsilon^{-j/(m-j)} \int_{\Omega} |u|^p dx \right).$$

**Proposition 3.7.** *The function  $\lambda : \mathbb{A} \rightarrow (0, \infty)$  is continuous and satisfies*

$$\Lambda := \sup_{\xi \in \mathbb{A}} \lambda(\xi) \leq \frac{g[\varrho]}{4\mu_-}. \quad (3.31)$$

Moreover,

$$\lim_{\xi \in \mathbb{A}, \xi \rightarrow 0} \lambda(\xi) = 0, \quad (3.32)$$

and if  $\kappa > 0$ , then also

$$\lim_{\xi \in \mathbb{A}, \xi \rightarrow \xi_c} \lambda(\xi) = 0 \quad \text{for any } \xi_c \in \bar{\mathbb{A}} \text{ with } |\xi_c| = |\xi|_c, \quad (3.33)$$

where  $\bar{\mathbb{A}}$  denotes the closure of  $\mathbb{A}$  and  $|\xi|_c$  is defined in (3.10).

PROOF. The proposition can be shown by a proof procedure similar to that used in [7, Proposition 3.9] but with necessary nontrivial modifications in arguments. For the reader's convenience, we give the proof in details here.

(i) We start with proving the boundedness of  $\lambda$ . By (3.24), we find that

$$\begin{aligned} -\lambda^2(\xi) &= \frac{1}{2} \int_{-1}^1 \lambda(\xi) \mu(4|\xi|^2 |\psi'_{\xi}|^2 + |\xi|^2 \psi_{\xi} + \psi_{\xi}''^2) \\ &\quad + |B|^2 \xi_1^2 (|\psi'_{\xi}|^2 + |\xi|^2 \psi_{\xi}^2) dx_3 - \frac{1}{2} |\xi|^2 (g[\rho] - \kappa |\xi|^2) \psi_{\xi}^2(0), \end{aligned}$$



which yields

$$2\mu_-|\xi|^2\lambda(\xi)\int_{-1}^1|\psi'_\xi|^2dx_3\leq\frac{1}{2}|\xi|^2g[\rho]\psi_\xi^2(0). \quad (3.34)$$

Using the Hölder inequality, we can bound

$$\psi_\xi^2(0)=\left|\int_0^1\psi'_\xi dx_3\right|^2\leq\int_0^1|\psi'_\xi|^2dx_3.$$

Substitution of the above inequality into (3.34) gives then

$$|\xi|^2\left(2\mu_-\lambda(|\xi|)-\frac{1}{2}g[\varrho]\right)\int_{-1}^1|\psi'_\xi|^2dx_3\leq 0. \quad (3.35)$$

Consequently, (3.35) implies (3.31), since  $\|\psi'_\xi\|_{L^2(-1,1)}>0$ .

(ii) We now turn to the proof of the continuity claim. Since  $\lambda=\sqrt{-\alpha}$ , it suffices to show the continuity of  $\alpha(\xi):=\alpha(|\xi_1|,|\xi_2|)$ . By virtue of Theorem 3.1, for each  $\xi\in\mathbb{A}$  there exists a function  $\psi_\xi\in\mathcal{A}$  satisfying (3.2)–(3.4), so that  $\alpha(\xi)=E(\psi_\xi)$ . Furthermore,  $\psi$  is smooth when restricted to  $(-1,0)$  or  $(0,1)$ . We have that  $\alpha(\xi)<0$ , which, when combined with (3.22), yields the estimate

$$\frac{1}{2}\int_{-1}^1s\mu||\xi|^2\psi_\xi+\psi_\xi''|^2dx_3-\frac{1}{2}|\xi|^2(g[\rho]-\kappa|\xi|^2)\psi_\xi^2(0)\leq\alpha(\xi)<0. \quad (3.36)$$

On the other hand,

$$|\xi|\psi_\xi^2(0)\leq 2\left(\int_0^1|\xi|^2\psi_\xi^2dx_3\right)^{1/2}\left(\int_0^1|\psi'_\xi|^2dx_3\right)^{1/2}\leq\frac{4}{\varrho_-} \quad (3.37)$$

because of  $\psi_\xi\in\mathcal{A}$ . Thus, plugging (3.37) into (3.36), we get

$$\frac{1}{2}\int_{-1}^1s\mu||\xi|^2\psi_\xi+\psi_\xi''|^2dx_3<2|\xi|(g[\rho]-\kappa|\xi|^2)/\varrho_-. \quad (3.38)$$

Now suppose  $\xi_n\in\mathbb{A}$  is a sequence so that  $\xi_n\rightarrow\xi\in\mathbb{A}$ . Since  $\mathbb{A}$  is a open set and  $\xi\neq(0,0)$ , when  $n$  is sufficiently large, there exists a sufficiently small, bounded and open sector domain  $\mathbb{S}\subset\mathbb{A}$  satisfying the following three conditions:

- there exists a  $n_0>0$  such that  $\xi,\xi_n\in\mathbb{S}$  for any  $n>n_0$ ;
- $(0,0)\notin\bar{\mathbb{S}}$ , where  $\bar{\mathbb{S}}$  denotes the closure of  $\mathbb{S}$ ;
- there is a  $\bar{\xi}\in\bar{\mathbb{S}}$  such that

$$\frac{|\xi_1|}{|\xi|}\leq\frac{|\bar{\xi}_1|}{|\bar{\xi}|}, \quad |\xi|\leq|\bar{\xi}| \quad \text{for any } \xi\in\mathbb{S}. \quad (3.39)$$

In order to make use of (3.38) we have to show that  $s(\xi)$  is bounded uniformly from below for  $n>n_0$ . Employing arguments similar to those used in the derivation of (3.29), we find that there is a  $\bar{\psi}\in H_0^2(-1,1)$ , such that  $\bar{\psi}\not\equiv 0$  and

$$\bar{E}_0(\bar{\psi}):=\frac{1}{2}\int_{-1}^1\frac{|\bar{\xi}_1B|^2}{|\bar{\xi}|^2}(|\bar{\psi}'|^2+|\bar{\xi}|^2\bar{\psi}^2)dx_3-\frac{1}{2}(g[\varrho]-\kappa|\bar{\xi}|^2)\bar{\psi}^2(0)<0,$$

which, together with (3.39), implies

$$\frac{1}{2} \int_{-1}^1 \frac{|\xi_1 B|^2}{|\xi|^2} (|\xi|^2 \bar{\psi}^2 + |\bar{\psi}'|^2) dx_3 - \frac{1}{2} (g[\varrho] - \kappa |\xi|^2) \bar{\psi}^2(0) < \bar{E}_0(\bar{\psi}) \quad \text{for any } \xi \in \mathbb{S}. \quad (3.40)$$

Let  $|\xi|_0 = \inf_{n > n_0} |\xi|$ , then  $|\xi|_0 > 0$ . Denoting

$$\begin{aligned} \bar{J}(\bar{\psi}) &= \frac{1}{2} \int_{-1}^1 \varrho (|\bar{\xi}|^2 \bar{\psi}^2 + |\bar{\psi}'|^2) dx_3, \quad J_c(\bar{\psi}) = \frac{1}{2} \int_{-1}^1 \varrho (|\xi|_c^2 \bar{\psi}^2 + |\bar{\psi}'|^2) dx_3, \\ \bar{E}_1(\bar{\psi}) &= \frac{1}{2} \int_{-1}^1 \mu (4|\bar{\xi}|^2 |\bar{\psi}'|^2 + 2|\bar{\xi}|^4 \bar{\psi}^2 + 2|\bar{\psi}''|^2) dx_3, \end{aligned}$$

and using (3.40), we argue, similarly to (3.30), to deduce that

$$\alpha(s) \leq -c_1 + s c_2 \quad \text{for any } \xi \in \mathbb{S},$$

where the two constants  $c_1 = c_1(|\xi|_0, \bar{E}_0(\bar{\psi}), \bar{J}(\bar{\psi}))$  and  $c_2 = c_2(\bar{E}_1(\bar{\psi}), J_c(\bar{\psi}))$  are independent of  $s$ . Keeping in mind that  $-\alpha(\xi_n) = \lambda^2(\xi_n) = s^2(\xi_n)$ , one gets

$$0 \leq s^2(\xi_n) + c_2 s(\xi_n) - c_1 \quad \text{for any } n > n_0.$$

On the other hand, the fact that  $\psi \in \mathcal{A}$  shows that  $\psi_{\xi_n}$  are uniformly bounded in  $H^1(-1, 1)$  for any  $n > n_0$ . Consequently, by virtue of (3.39) and (3.18), the estimate (3.38) implies the uniform boundedness of  $\psi_{\xi_n}$  in  $H_0^2(-1, 1)$  for any  $n > n_0$ .

Plugging the above boundedness facts on  $\psi_{\xi_n}$  into the ODE (3.5) in the intervals  $(-1, 0)$  and  $(0, 1)$  respectively, we find that  $\psi_{\xi_n}''''$  are also uniformly bounded in  $L^2(-1, 0)$  and  $L^2(0, 1)$  for any  $n > n_0$ . Thus, using Lemma 3.1, we infer that  $\psi_{\xi_n}$  are uniformly bounded in  $H^4(-1, 0)$  and  $H^4(0, 1)$  for any  $n > n_0$ . So, up to extraction of a subsequence we conclude that

$$\psi_{\xi_n} \rightarrow \psi \quad \text{strongly in } H_0^2(-1, 1), H^3(-1, 0) \text{ and } H^3(0, 1),$$

which yields

$$\alpha(\xi_n) = E(\psi_{\xi_n}) \rightarrow E(\psi_\xi) = \alpha(\xi). \quad (3.41)$$

Since (3.41) must hold for any such extracted subsequence, one deduces that  $\alpha(\xi_n) \rightarrow \alpha(\xi)$  for the original sequence  $\xi_n$  as well, and hence  $\alpha(\xi)$  is continuous.

Finally, we derive the limits as  $\xi \rightarrow 0$  when  $\kappa \geq 0$ , and  $\xi \rightarrow \xi_c \in \bar{\mathbb{A}}$  with  $|\xi_c| = |\xi|_c$  when  $\kappa > 0$ , where we restrict the variables  $\xi$  in  $\mathbb{A}$ . By virtue of (3.24) and (3.37),  $0 \leq \lambda^2(\xi) \leq 2g[\varrho]|\xi|/\varrho_-$ , which gives (3.32). On the other hand, when  $\kappa > 0$  we may utilize (3.24) to find that

$$\lambda^2(\xi) \leq (g[\varrho] - \kappa |\xi|^2) |\xi|^2 \psi_\xi^2(0)/2 \leq 2|\xi|(g[\varrho] - \kappa |\xi|^2)/\varrho_-,$$

which results in (3.33) when  $\kappa > 0$ . □

**Remark 3.2.** The stabilizing effect of viscosity and surface tension is evident from the above calculations. As shown in [3], without viscosity or surface tension, there exists a domain  $\mathbb{D}$ , such that  $\lambda(\xi) \rightarrow \infty$  for  $\xi \in \mathbb{D}$  as  $|\xi| \rightarrow \infty$ . With viscosity but no surface tension, by virtue of the definition (3.13), there are only partial spatial frequencies  $\xi$  (including all  $\xi$  with  $\xi_1 = 0$  in particular) which remain unstable, but the growth of  $\lambda(\xi)$  is bounded. With both viscosity and surface tension, only those spatial frequencies  $\xi$  belonging to the bounded domain  $\mathbb{A}$  are unstable, and  $\lambda(\xi)$  remains bounded. In addition, in the construction of the normal mode solution to the linearized system, the horizontal magnetic field can enhance the maximal growth rate  $\lambda(\xi)$ . In particular, when  $\xi_1 = 0$ , the growth rate reduces to the one for the corresponding equations of incompressible viscous fluids.

**Remark 3.3.** From Proposition 3.7, we immediately infer that

$$\lambda_0 := \inf_{\xi \in \mathbb{D}} \lambda(\xi) > 0, \quad (3.42)$$

where the closed domain  $\mathbb{D}$  is defined by (3.14).

### 3.3. Construction of a solution to the system (3.2)–(3.4)

A solution to (3.5)–(3.9) gives rise to a solution of the system (3.2)–(3.4) for the growing mode velocity  $v$ , as well.

**Theorem 3.2.** *For each  $\xi \in \mathbb{A}$ , there exists a solution  $(\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \tilde{\pi}) = (\tilde{\varphi}(\xi, x_3), \tilde{\theta}(\xi, x_3), \tilde{\psi}(\xi, x_3), \tilde{\pi}(\xi, x_3))$  with  $\lambda = \lambda(|\xi_1|, |\xi_2|) > 0$  to (3.2)–(3.4), and the solution is smooth when restricted to  $(-1, 0)$  or  $(0, 1)$ . Moreover,*

$$\|\tilde{\varphi}\|_{L^2(-1,1)}^2 + \|\tilde{\theta}\|_{L^2(-1,1)}^2 + \|\tilde{\psi}\|_{L^2(-1,1)}^2 = 1, \quad (3.43)$$

$$\|\tilde{\psi}'\|_{L^2(-1,1)} \leq |\xi| \sqrt{2\varrho_+/\varrho_-}. \quad (3.44)$$

PROOF. In view of Theorem 3.1, we have a solution  $(\psi, \lambda) := (\psi(|\xi_1|, |\xi_2|, x_3), \lambda(|\xi_1|, |\xi_2|))$  satisfying (3.5)–(3.9). Moreover,  $\lambda > 0$  and  $\psi \in \mathbb{A}$  is smooth when restricted to  $(-1, 0)$  or  $(0, 1)$ . Then, multiplying (3.2)<sub>1</sub> and (3.2)<sub>2</sub> by  $\xi_1$  and  $\xi_2$  respectively, adding the resulting equations, and utilizing (3.2)<sub>4</sub>, we find that  $\pi$  can be expressed by  $\psi$ , i.e.,

$$\pi = \pi(|\xi_1|, |\xi_2|, x_3) = [\lambda\mu\psi''' - (\lambda^2\varrho + \lambda\mu|\xi|^2 + B^2\xi_1^2)\psi']/(\lambda|\xi|^2). \quad (3.45)$$

Notice that (3.2)<sub>1</sub> can be rewritten as

$$\varphi'' - (\lambda^2\varrho + \lambda\mu|\xi|^2 + B^2\xi_1^2)\varphi/(\lambda\mu) = -\xi_1\pi/\mu \quad (3.46)$$

with jump and boundary conditions

$$[\varphi] = 0, \quad [\mu(\xi_1\psi - \varphi')] = 0, \quad \varphi(-1) = \varphi(1) = 0. \quad (3.47)$$

Hence, we can easily construct a unique solution of the form

$$\varphi = (\xi, x_3) = \begin{cases} \xi_1(c_1 e^{a_+ x_3} + c_2 e^{-a_+ x_3} - f_+(x_3)), & \text{on } (0, 1), \\ \xi_1(c_3 e^{a_- x_3} + c_4 e^{-a_- x_3} - f_-(x_3)), & \text{on } (-1, 0) \end{cases} \quad (3.48)$$

to the equation (3.46) with jump and boundary conditions (3.47), where

$$a_{\pm} = \sqrt{|\xi|^2 + \frac{\lambda\varrho}{\mu_{\pm}} + \frac{B^2\xi_1^2}{\lambda\mu_{\pm}}},$$

$$f_{\pm}(x_3) = \frac{1}{2a_{\pm}\mu_{\pm}} \int_0^{x_3} \pi(e^{a_{\pm}(x_3-y)} - e^{a_{\pm}(y-x_3)}) dy,$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ \mu_+ a_+ & -\mu_+ a_+ & -\mu_- a_- & \mu_- a_- \\ e^{a_+} & e^{-a_+} & 0 & 0 \\ 0 & 0 & e^{-a_-} & e^{a_-} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ (\mu_+ - \mu_-)\psi(0) \\ f(1) \\ f(-1) \end{bmatrix}.$$

Similarly to (3.48),

$$\theta := \theta(\xi, x_3) = \begin{cases} \xi_2(c_1 e^{a+x_3} + c_2 e^{-a+x_3} - f_+(x_3)), & \text{on } (0, 1), \\ \xi_2(c_3 e^{a-x_3} + c_4 e^{-a-x_3} - f_-(x_3)), & \text{on } (-1, 0) \end{cases} \quad (3.49)$$

is a unique solution of (3.2)<sub>2</sub> with jump and boundary conditions:

$$[\![\theta]\!] = 0, \quad [\![\mu(\xi_2 \psi - \theta')]\!] = 0, \quad \theta(-1) = \theta(1) = 0.$$

Consequently,  $(\varphi, \theta, \psi, \pi)$  is a solution to the system (3.2)–(3.4). Now, we define

$$\begin{aligned} (\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \tilde{\pi}) &:= (\tilde{\varphi}(\xi, x_3), \tilde{\theta}(\xi, x_3), \tilde{\psi}(\xi, x_3), \tilde{\pi}(\xi, x_3)) \\ &= (\varphi, \theta, \psi, \pi) / (\|\varphi\|_{L^2(-1,1)}^2 + \|\theta\|_{L^2(-1,1)}^2 + \|\psi\|_{L^2(-1,1)}^2). \end{aligned}$$

Thus,  $(\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \tilde{\pi})$  is still a solution to the system (3.2)–(3.4), and moreover,  $(\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \tilde{\pi})$  satisfies (3.43).

Finally, making use of (3.2)<sub>4</sub> and (3.23), we conclude that

$$\begin{aligned} \frac{1}{\varrho_+ |\xi|^2} &= \frac{1}{2\varrho_+ |\xi|^2} \int_{-1}^1 \varrho(|\xi|^2 |\psi|^2 + |\psi'|^2) dx_3 \\ &\leq \int_{-1}^1 (|\varphi|^2 + |\theta|^2 + |\psi|^2) dx_3 \\ &= \|\varphi\|_{L^2(-1,1)}^2 + \|\theta\|_{L^2(-1,1)}^2 + \|\psi\|_{L^2(-1,1)}^2 \end{aligned}$$

and

$$\int_{-1}^1 |\psi'|^2 dx_3 \leq 2/\varrho_-.$$

The above two inequalities imply (3.44) immediately.  $\square$

**Remark 3.4.** For each  $x_3$ , it is easy to see that the solution  $(\tilde{\varphi}(\xi, \cdot), \tilde{\theta}(\xi, \cdot), \tilde{\psi}(\xi, \cdot), \tilde{\pi}(\xi, \cdot), \lambda(|\xi|))$  constructed in Theorem 3.2 possesses the following properties:

- (1)  $\lambda(|\xi_1|, |\xi_2|)$ ,  $\tilde{\psi}(\xi, \cdot)$  and  $\tilde{\pi}(\xi, \cdot)$  are even on  $\xi_1$  or  $\xi_2$ , when the other variable is fixed;
- (2)  $\tilde{\varphi}(\xi, \cdot)$  is odd on  $\xi_1$ , but even on  $\xi_2$ , when the other variable is fixed;
- (3)  $\tilde{\theta}(\xi, \cdot)$  is even on  $\xi_1$ , but odd on  $\xi_2$ , when the other variable is fixed.

The next result provides an estimate for the  $H^k$ -norm of the solutions  $(\varphi, \theta, \psi, \pi)$  with  $\xi$  varying in domain  $\mathbb{D}$ , which will be useful in the next section when such solutions are integrated in a Fourier synthesis. To emphasize the dependence on  $\xi$ , we write these solutions as  $(\tilde{\varphi}(\xi) = \tilde{\varphi}(\xi, x_3), \tilde{\theta}(\xi) = \tilde{\theta}(\xi, x_3), \tilde{\psi}(\xi) = \tilde{\psi}(\xi, x_3), \tilde{\pi}(\xi) = \tilde{\pi}(\xi, x_3))$ .

**Lemma 3.2.** *Let  $\xi \in \mathbb{D}$ ,  $\theta(\xi) := \tilde{\theta}(\xi)$ ,  $\psi(\xi) := \tilde{\psi}(\xi)$ ,  $\pi(\xi) := \tilde{\pi}(\xi)$  and  $\lambda(|\xi_1|, |\xi_2|)$  be constructed in Theorem 3.2, then for any  $k \geq 0$  there exist positive constants  $a_k$ ,  $b_k$  and  $c_k$  depending on  $d_1$ ,  $d_2$ ,  $\lambda_0$ ,  $|B|$ ,  $\varrho_{\pm}$ ,  $\mu_{\pm}$  and  $g$ , so that*

$$\|\varphi(\xi)\|_{H^k(-1,0)} + \|\varphi(\xi)\|_{H^k(0,1)} + \|\theta(\xi)\|_{H^k(-1,0)} + \|\theta(\xi)\|_{H^k(0,1)} \leq a_k, \quad (3.50)$$

$$\|\psi(\xi)\|_{H^k(-1,0)} + \|\psi(\xi)\|_{H^k(0,1)} \leq b_k, \quad (3.51)$$

$$\|\pi(\xi)\|_{H^k(-1,0)} + \|\pi(\xi)\|_{H^k(0,1)} \leq c_k, \quad (3.52)$$

where  $d_1$  and  $d_2$  are the same constants in (3.14), and  $\lambda_0$  is defined by (3.42). Moreover,

$$\|\varphi\|_{L^2(-1,1)}^2 + \|\theta\|_{L^2(-1,1)}^2 + \|\psi\|_{L^2(-1,1)}^2 = 1. \quad (3.53)$$

PROOF. Throughout this proof, we denote by  $\tilde{c}_1, \dots, \tilde{c}_8$  generic positive constants which may depend on  $d_1, d_2, \lambda_0, |B|, \varrho_\pm, \mu_\pm$  and  $g$ , but not on  $|\xi|$ . Obviously, (3.53) follows from (3.44) immediately.

(i) First we write (3.5) as

$$\psi''''(\xi) = [(\lambda^2 \varrho + 2\lambda\mu|\xi|^2 + B^2\xi_1^2)\psi''(\xi) - |\xi|^2(\lambda^2 \varrho + \lambda\mu|\xi|^2 + B^2\xi_1^2)\psi(\xi)]/(\lambda\mu). \quad (3.54)$$

If we make use of (3.31), (3.42), and the fact  $|\xi| \leq d_2$ , Lemma 3.1 and the Cauchy-Schwarz inequality, we see that there exists two constants  $\tilde{c}_1$  and  $\tilde{c}_2$ , such that

$$\begin{aligned} \|\psi''''(\xi)\|_{L^2(I_\pm)} &\leq \tilde{c}_1 (\|\psi(\xi)\|_{L^2(I_\pm)} + \|\psi''(\xi)\|_{L^2(I_\pm)}) \\ &\leq (\tilde{c}_2 + 1)(\varepsilon^{-1/2}\|\psi(\xi)\|_{L^2(I_\pm)} + \sqrt{\varepsilon}\|\psi''''(\xi)\|_{L^2(I_\pm)}) \quad \text{for any } \varepsilon \in (0, 1), \end{aligned} \quad (3.55)$$

respectively, where  $I_+ = (0, 1)$  and  $I_- = (-1, 0)$ . Choosing  $\sqrt{\varepsilon} = 1/\{2(\tilde{c}_2 + 1)\}$  in (3.55) and using (3.53), we arrive at

$$\|\psi''''(\xi)\|_{L^2(I_\pm)} \leq \tilde{c}_3 \quad \text{for some constant } \tilde{c}_3 > 0, \quad (3.56)$$

whence,

$$\|\psi''''(\xi)\|_{L^2(-1,1)} \leq \tilde{c}_4 := 2\tilde{c}_3. \quad (3.57)$$

Thus from (3.56) and Lemma 3.1 we get

$$\|\psi''(\xi)\|_{L^2(-1,1)} \leq \tilde{c}_5. \quad (3.58)$$

and

$$\|\psi''''(\xi)\|_{L^2(-1,1)} \leq \tilde{c}_6. \quad (3.59)$$

Summarizing the estimates (3.53), (3.44), and (3.57)–(3.59), we conclude that, for each non-negative integer  $k \in [0, 4]$ , there exists a constant  $\tilde{b}_k > 0$  depending on  $d_2, |B|, \lambda_0, \varrho_\pm, \mu_\pm$  and  $g$ , such that

$$\|\psi^{(k)}(\xi)\|_{L^2(-1,1)} \leq \tilde{b}_k. \quad (3.60)$$

Differentiating (3.54) with respect to  $x_3$  and using (3.60), we find, by induction on  $k$ , that (3.60) holds for any  $k \geq 0$ . This gives (3.51).

(ii) Recalling the expression (3.45) of  $\pi$  and the fact that  $|\xi| \geq d_1$ , we employ (3.42) and (3.31) to deduce that for any  $k \geq 0$ ,

$$\begin{aligned} \|\pi^{(k)}(\xi)\|_{L^2(-1,1)} &\leq \frac{\mu^+}{|\xi|^2} \|\psi^{(k+3)}(\xi)\|_{L^2(-1,1)} + \left( \frac{\lambda\varrho^+}{|\xi|^2} + \mu^+ + \frac{B^2\xi_1^2}{\lambda|\xi|^2} \right) \|\psi^{(k+1)}(\xi)\|_{L^2(-1,1)} \\ &\leq \frac{\mu^+}{d_1^2} \tilde{b}_{k+3} + \left( \frac{\Lambda\varrho^+}{d_1^2} + \mu^+ + \frac{B^2}{\lambda_0} \right) \tilde{b}_{k+1}, \end{aligned}$$

which implies (3.52).

(iii) Making use of (3.46), (3.52), (3.53), (3.31) and  $|\xi| \leq d_2$ , we get

$$\|\varphi''(\xi)\|_{L^2(-1,1)} \leq \frac{d_2}{\mu_-} \|\pi(\xi)\|_{L^2(-1,1)} + \left( \frac{\Lambda\varrho^+}{\mu_-} + d_2^2 + \frac{B^2d_2^2}{\lambda_0\mu_-} \right) \|\varphi(\xi)\|_{L^2(-1,1)} \leq \tilde{c}_7. \quad (3.61)$$

Applying (3.61), (3.53) and Lemma 3.1, we obtain

$$\|\varphi'(\xi)\|_{L^2(-1,0)} + \|\varphi'(\xi)\|_{L^2(0,1)} \leq \tilde{c}_8. \quad (3.62)$$

Combining (3.53) with (3.38) and (3.62), we conclude that, for each nonnegative integer  $k \in [0, 2]$ , there exists a constant  $\tilde{a}_k > 0$  depending on  $d_1, d_2, \lambda_0, \varrho_{\pm}, \mu_{\pm}$  and  $g$ , so that

$$\|\varphi^{(k)}(\xi)\|_{L^2(-1,1)} \leq \tilde{a}_k. \quad (3.63)$$

Thus, by virtue of (3.46) and induction on  $k$ , (3.63) holds for any  $k \geq 0$ . Following the same procedure as used in estimating  $\varphi$ , we infer that for each  $k \geq 0$ ,

$$\|\theta^{(k)}(\xi)\|_{L^2(-1,1)} \leq \tilde{d}_k \quad (3.64)$$

for some constant  $\tilde{d}_k$  depending on  $d_1, d_2, \lambda_0, \varrho_{\pm}, \mu_{\pm}$  and  $g$ . Adding (3.64) to (3.63), we arrive at

$$\|\varphi^{(k)}(\xi)\|_{L^2(-1,1)} + \|\theta^{(k)}(\xi)\|_{L^2(-1,1)} \leq (\tilde{a}_k + \tilde{d}_k) \text{ for any } k \geq 0,$$

which yields (3.50). This completes the proof.  $\square$

### 3.4. Fourier synthesis

In this section we will use the Fourier synthesis to build growing solutions to (1.20) with  $\bar{B} = (B, 0, 0)$  out of the solutions constructed in the previous section (Theorem 3.2) for fixed spatial frequency  $\xi \in \mathbb{A}$ . The constructed solutions will grow in-time in the piecewise Sobolev space of order  $k$ ,  $\bar{H}^k(\Omega)$ , defined by (2.1).

**Theorem 3.3.** *Let  $\bar{B} = (B, 0, 0)$  and  $f \in C_0^\infty[0, \infty)$  be a real-valued function. For any  $\xi \in \mathbb{R}^2$ , we define*

$$w(\xi, x_3) = -i\tilde{\varphi}(\xi, x_3)e_1 - i\tilde{\theta}(\xi, x_3)e_2 + \tilde{\psi}(\xi, x_3)e_3,$$

where

$$(\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \tilde{\pi})(\xi, x_3) = \begin{cases} \text{the solutions provided by Theorem 3.2} \\ \text{with } \lambda(|\xi_1|, |\xi_2|) > 0, & \text{if } \xi \in \mathbb{D}, \\ \text{zero, and } \lambda(|\xi_1|, |\xi_2|) \equiv \lambda_0, & \text{if } \xi \notin \mathbb{D}. \end{cases}$$

Denote

$$\eta(t, x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(|\xi|) w(\xi, x_3) e^{\lambda(|\xi_1|, |\xi_2|)t} e^{ix'\xi} d\xi, \quad (3.65)$$

$$v(t, x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(|\xi_1|, |\xi_2|) f(|\xi|) w(\xi, x_3) e^{\lambda(|\xi_1|, |\xi_2|)t} e^{ix'\xi} d\xi, \quad (3.66)$$

$$q(t, x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(|\xi_1|, |\xi_2|) f(|\xi|) \tilde{\pi}(\xi, x_3) e^{\lambda(|\xi_1|, |\xi_2|)t} e^{ix'\xi} d\xi. \quad (3.67)$$

Then,  $(\eta, v, q)$  is a real-valued solution to the linearized problem (1.20) along with jump and boundary conditions (1.21) and (1.22). For every  $k \in \mathbb{N}$ , we have the estimate

$$\|\eta(0)\|_{\bar{H}^k} + \|v(0)\|_{\bar{H}^k} + \|q(0)\|_{\bar{H}^k} \leq \tilde{c}_k \left( \int_{\mathbb{R}^2} f^2(|\xi|) d\xi \right)^{1/2} < \infty, \quad (3.68)$$

where  $\tilde{c}_k > 0$  is a constant depending on the parameters  $d_1, d_2, |B|, \lambda_0, \varrho_{\pm}, \mu_{\pm}$  and  $g$ . Moreover, for every  $t > 0$  we have  $\eta(t), v(t), q(t) \in \bar{H}^k(\Omega)$ , and

$$e^{t\lambda_0} \|\eta(0)\|_{\bar{H}^k} \leq \|\eta(t)\|_{\bar{H}^k} \leq e^{t\Lambda} \|\eta(0)\|_{\bar{H}^k}, \quad (3.69)$$

$$e^{t\lambda_0} \|v(0)\|_{\bar{H}^k} \leq \|v(t)\|_{\bar{H}^k} \leq e^{t\Lambda} \|v(0)\|_{\bar{H}^k}, \quad (3.70)$$

$$e^{t\lambda_0} \|q(0)\|_{\bar{H}^k} \leq \|q(t)\|_{\bar{H}^k} \leq e^{t\Lambda} \|q(0)\|_{\bar{H}^k}, \quad (3.71)$$

where  $\lambda_0$  and  $\Lambda$  are defined in (3.42) and (3.31), respectively.

PROOF. For each fixed  $\xi \in \mathbb{R}^2$ ,

$$\begin{aligned}\eta(t, x') &= f(|\xi|)w(\xi, x_3)e^{\lambda(|\xi_1|, |\xi_2|)t}e^{ix'\xi}, \\ v(t, x) &= \lambda(|\xi_1|, |\xi_2|)f(|\xi|)w(\xi, x_3)e^{\lambda(|\xi_1|, |\xi_2|)t}e^{ix'\xi} \\ q(t, x) &= \lambda(|\xi_1|, |\xi_2|)f(|\xi|)\pi(\xi, x_3)e^{\lambda(|\xi_1|, |\xi_2|)t}e^{ix'\xi}\end{aligned}$$

give a solution to (1.20). Since  $f \in C_0^\infty[0, \infty)$ , by the construction of  $w$ , Lemma 3.2 implies that

$$\sup_{\xi \in \text{supp}(f)} \|\partial_3^k w(\xi, \cdot)\|_{L^\infty(\Omega)} < \infty \quad \text{for all } k \in \mathbb{N}.$$

Also,  $\lambda(\xi) \leq \Lambda$ . These bounds show that the Fourier synthesis of the solution given by (3.65)–(3.67) is also a solution of (1.20). Because  $f$  is real-valued and radial,  $\mathbb{D}$  is a symmetrical domain (see Proposition 3.1), we can easily verify, recalling Remark 3.4, that the Fourier synthesis is real-valued.

The estimate (3.68) follows from Lemma 3.2 with arbitrary  $k \geq 0$  and the construction of  $(\eta, v, q)$ . Finally, we can use (3.42), (3.31) and (3.65)–(3.67) to obtain the estimates (3.69)–(3.71).  $\square$

## 4. Global instability for the linearized problem

### 4.1. Uniqueness of the linearized equations

In this section, we will show the uniqueness of solutions to the linearized problem with lower regularity, which will be applied to the proof of Theorem 2.2 in Section 5. For this purpose, we need a generalized formula of integrating by parts (or Gauss-Green's formula). Let us first recall the boundary trace theorem (see Theorem 5.36 in [1, Chapter 5]).

**Lemma 4.1.** *Let  $U$  be a domain in  $\mathbb{R}^n$  satisfying the uniform  $C^m$ -regularity condition, and assume that there exists a simple  $(m, p)$ -extension operator  $E$  for  $U$ . Also assume that  $mp < n$  and  $p \leq q \leq p^* = (n-1)p/(n-mp)$ . Then, there exists a bounded linear operator*

$$\gamma^U : W^{m,p}(U) \rightarrow L^q(\partial U),$$

such that

$$\gamma^U(u) = u \text{ on } \partial U$$

for all  $u \in W^{m,p}(U) \cap C(\bar{U})$ .

The function  $\gamma^U(u) \in L^q(\partial U)$  is called the trace of the function of  $u \in W^{1,p}(U)$  on the boundary  $\partial U$ . By the Stein extension theorem (see Theorem 5.24 in [1, Chapter 5]) and the definition of the uniform  $C^m$ -regularity condition (see Definition 4.10 in [1, Chapter 5]), it is easy to verify that  $\Omega$ ,  $\Omega_+$  and  $\Omega_-$  have different simple  $(m, p)$ -extension operators. Keeping these facts in mind, we can start to show the following formula of integrating by parts. For convenience in the subsequent analysis, we will use the notations  $\gamma_+(f) := \gamma^{\Omega_+}(f_+)$  and  $\gamma_-(f) := \gamma^{\Omega_-}(f_-)$ .

**Lemma 4.2.** *For all  $v \in H_0^1(\Omega)$  and  $w \in \bar{H}^1(\Omega)$ , we have*

$$\int_{\Omega} \partial_i w v dx = - \int_{\Omega} w \partial_i v dx + \int_{\mathbb{R}^2} (\gamma_+(w) - \gamma_-(w)) \gamma_+(v) \alpha_i dx \quad (4.1)$$

for  $i = 1, 2, 3$ , where  $\alpha_1 = \alpha_2 = 0$  and  $\alpha_3 = -1$ .

PROOF. Temporarily suppose  $\bar{v} \in C_0^1(\Omega)$ ,  $\bar{w}_+ \in C^1(\bar{\Omega}_+)$  and  $\bar{w}_- \in C^1(\bar{\Omega}_-)$ . By the Gauss-Green theorem, we have

$$\int_{\Omega} \partial_i \bar{w} \bar{v} dx = - \int_{\Omega} \bar{w} \partial_i \bar{v} dx + \int_{\mathbb{R}^2} ((\bar{w}_+ - \bar{w}_-) \bar{v})(x', 0) \alpha_i dx. \quad (4.2)$$

Using Lemma 4.1, one has

$$\begin{aligned} \|(\bar{v} - \gamma_+(v))(x', 0)\|_{L^2(\mathbb{R}^2)} &\leq \|\bar{v} - \gamma_+(v)\|_{L^2(\partial\Omega_+)} \\ &= \|\gamma_+(\bar{v} - v)\|_{L^2(\partial\Omega_+)} \leq c \|\bar{v} - v\|_{H^1(\Omega_+)} \\ &\leq c \|\bar{v} - v\|_{H_0^1(\Omega)} \end{aligned}$$

and

$$\|(\bar{w}_+ - \gamma_+(w_+))(x', 0)\|_{L^2(\mathbb{R}^2)} \leq c \|\bar{w}_+ - w_+\|_{H^1(\Omega_+)}$$

for some constant  $c > 0$ . By the Hölder inequality, the above two estimates imply that

$$\begin{aligned} &\|(\bar{w}_+ \bar{v} - \gamma_+(w) \gamma_+(v))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|(\bar{v}(\bar{w}_+ - \gamma_+(w)))(x', 0)\|_{L^1(\mathbb{R}^2)} + \|(\gamma_+(w)(\bar{v} - \gamma_+(v)))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|\bar{v}(x', 0)\|_{L^2(\mathbb{R}^2)} \|(\bar{w}_+ - \gamma_+(w))(x', 0)\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|\gamma_+(w)(x', 0)\|_{L^2(\mathbb{R}^2)} \|(\bar{v} - \gamma_+(v))(x', 0)\|_{L^2(\mathbb{R}^2)} \\ &\leq c^2 \|\bar{v}\|_{H_0^1(\Omega)} \|\bar{w}_+ - w_+\|_{H^1(\Omega_+)} + c^2 \|w_+\|_{H^1(\Omega_+)} \|\bar{v} - v\|_{H_0^1(\Omega)}. \end{aligned} \quad (4.3)$$

Similarly to (4.3), one gets

$$\begin{aligned} &\|(\bar{w}_- \bar{v} - \gamma_-(w) \gamma_-(v))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq c^2 (\|\bar{v}\|_{H_0^1(\Omega)} \|\bar{w}_- - w_-\|_{H^1(\Omega_-)} + \|w_-\|_{H^1(\Omega_-)} \|\bar{v} - v\|_{H_0^1(\Omega)}). \end{aligned} \quad (4.4)$$

In addition, if  $\bar{v}_m \rightarrow v$  strongly in  $H_0^1(\Omega)$ , then there exists  $m_0 > 0$  such that

$$\|\bar{v}_m\|_{H_0^1(\Omega)} \leq \|v\|_{H_0^1(\Omega)} + 1 \quad \text{for any } m \geq m_0. \quad (4.5)$$

Since  $C_0(\Omega)$  is dense in  $H_0^1(\Omega)$  and  $C_0(\mathbb{R}^3)$  dense in  $H^1(\Omega_{\pm})$ , the identity (4.1) follows from (4.2)–(4.5) and a standard density argument.  $\square$

**Definition 4.1.** Given  $T > 0$  and the initial data  $(\eta_0, v_0)$  to the linearized problem (1.20)–(1.22), a triple  $(\eta, v, q)$  is called a strong solution of (1.20)–(1.22), if

(1)  $\eta, v \in C^0([0, T], L^2(\Omega))$ ,  $\eta(0) = \eta_0$ ,  $v(0) = v_0$  and

$$\text{ess sup}_{0 < t < T} (\|v(t)\|_{\bar{H}^3} + \|\eta(t)\|_{\bar{H}^3} + \|q(t)\|_{\bar{H}^1} + \|v(t)\|_{H_0^1(\Omega)}) < \infty. \quad (4.6)$$

(2) The equations

$$\partial_t \eta = v, \quad (4.7)$$

$$\varrho \partial_t v + \nabla q = \mu \Delta v + \sum_{1 \leq l, m \leq 3} \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta, \quad (4.8)$$

$$\text{div } v = 0 \quad (4.9)$$

hold a.e. in  $(0, T] \times (\Omega \setminus \{x_3 = 0\})$ .



(3) For a.e.  $t \in (0, T)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^2} (\gamma_+(q)I - \mu_+(\nabla v_+ + \nabla v_+^T) - (\gamma_-(q)I - \mu_-(\nabla v_- + \nabla v_-^T))) e_3 \cdot \varphi dx' \\ &= \int_{\mathbb{R}^2} \left( g[\varrho] \eta_{+3} \varphi_3 + \sum_{l=1}^3 \bar{B}_3 \bar{B}_l (\partial_l \eta_+ - \partial_l \eta_-) \cdot \varphi - \kappa \sum_{i=1}^2 \partial_i \eta_{+3} \partial_i \varphi_3 \right) dx' \end{aligned} \quad (4.10)$$

holds for any  $\varphi \in H_0^1(\mathbb{R}^2)$ , where  $\eta_{+3}$  and  $v_3$  are the third component of  $\eta_+$  and  $v$ , respectively.

**Remark 4.1.** Since  $v(t) \in H_0^1(\Omega) \cap \bar{H}^3(\Omega)$  for each  $t \geq 0$ , we can make use of the embedding theorem and (4.9) to obtain

$$v(t) \in C^0(\bar{\Omega}), \quad v_+(t) \in C^1(\bar{\Omega}_+), \quad v_-(t) \in C^1(\bar{\Omega}_-), \quad (4.11)$$

$$v(t) \equiv 0 \text{ on } \partial\Omega, \quad (4.12)$$

$$\nabla_{x'} v_+ \equiv \nabla_{x'} v_- \text{ on } \mathbb{R}^2, \quad (4.13)$$

$$\operatorname{div} v(t) \equiv 0 \quad \text{in } \bar{\Omega} \quad \text{for a.e. } t \geq 0.$$

Thus, in view of (4.13), we define for the sake of simplicity that

$$\nabla_{x'} v := \nabla_{x'} v_+ = \nabla_{x'} v_- \quad \text{on } \mathbb{R}^2 \times \{0\}.$$

Moreover, by virtue of Lemma 4.1, there is a constant  $c$  such that

$$\|v(t, x', 0)\|_{H^1(\mathbb{R}^2)} \leq c \|v(t)\|_{H^2(\Omega_{\pm})} \quad \text{for a.e. } t \geq 0. \quad (4.14)$$

**Remark 4.2.** In view the regularity of  $(\eta, v, q)$  in Definition 4.1, we can see that the equality (4.10) makes sense.

**Remark 4.3.** It is easy to verify that any  $(\eta, v, q)$ , which is a solution established in Theorem 3.3, is a strong solution to the linearized system (1.20)–(1.22).

**Theorem 4.1.** (Uniqueness) *Let  $\bar{B} := (\bar{B}_1, \bar{B}_2, \bar{B}_3)$  be a constant vector. Assume that  $(\tilde{\eta}, \tilde{v}, \tilde{q})$  and  $(\bar{\eta}, \bar{v}, \bar{q})$  are two strong solutions to (1.20)–(1.22), with  $\tilde{v}(0) = \bar{v}(0) = v_0$ ,  $\tilde{\eta}(0) = \bar{\eta}(0) = \eta_0$ . Then  $(\tilde{\eta}, \tilde{v}, \nabla \tilde{q}) = (\bar{\eta}, \bar{v}, \nabla \bar{q})$ .*

PROOF. Let  $(\eta, v, q) = (\tilde{\eta} - \bar{\eta}, \tilde{v} - \bar{v}, \tilde{q} - \bar{q})$ . Recalling Definition 4.1,  $(\eta, v, q)$  is still a strong solution to the linearized system (1.20)–(1.22) with zero initial data, i.e.,  $\eta(0) = 0$  and  $v(0) = 0$ .

Let  $t_1 \in (0, T]$ . Multiplying (4.8) by  $v$ , integrating over  $(0, \tau) \times \Omega$  for any  $\tau \in (0, t_1)$  and using (4.8), we find that

$$\begin{aligned} & \int_0^\tau \int_\Omega \varrho \partial_t v \cdot v dx dt + \int_0^\tau \int_\Omega \operatorname{div}(qI - \mu(\nabla v + \nabla v^T)) \cdot v dx dt \\ &= \sum_{1 \leq l, m \leq 3} \int_0^\tau \int_\Omega \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta \cdot v dx dt. \end{aligned} \quad (4.15)$$

(1) Firstly, we transform (4.15) to the form of energy equality. By virtue of the regularity (4.6), (4.8) implies that

$$\partial_t v \in L^2((0, T) \times \Omega),$$

which, together with  $v \in L^\infty(0, T; H^1(\Omega)) \cap C^0([0, T], L^2(\Omega))$ , yields

$$\int_0^\tau \int_\Omega \varrho \partial_t v \cdot v dx dt = \frac{1}{2} \int_\Omega \varrho v^2(\tau) dx - \frac{1}{2} \int_\Omega \varrho v^2(0) dx. \quad (4.16)$$

Thanks to Lemma 4.1, (4.11)–(4.12), and regularity of  $q$ , we obtain

$$\begin{aligned} & \int_0^\tau \int_\Omega \operatorname{div}(qI - \mu(\nabla v + \nabla v^T)) \cdot v dx dt \\ &= \int_0^\tau \int_{\mathbb{R}^2} (\gamma_-(q)I - \mu_-(\nabla v_- + \nabla v_-^T) - (\gamma_+(q)I - \mu_+(\nabla v_+ + \nabla v_+^T))) e_3 \cdot v dx' dt \\ &+ \int_0^\tau \int_\Omega \mu \nabla v : (\nabla v + \nabla v^T) dx dt. \end{aligned} \quad (4.17)$$

Employing (4.14),  $\operatorname{div} v = 0$  at the plan  $\{x_3 = 0\}$  and the arguments similar to those used for (4.1), we can show

$$\sum_{j=1}^3 \int_0^\tau \int_{\mathbb{R}^2} \mu v_j \partial_j v_{\pm 3} dx' dt = 0.$$

Using the above equality, Lemma 4.1, and the regularity of  $u$  stated in Remark 4.1, we conclude that

$$\int_0^\tau \int_\Omega \mu \nabla v : \nabla v^T dx dt = \sum_{1 \leq i, j \leq 3} \int_0^\tau \int_\Omega \mu \partial_i v_j \partial_j v_i dx dt \equiv 0. \quad (4.18)$$

In view of (4.15)–(4.18) and  $v(0) = 0$ , we find the following energy equality

$$\begin{aligned} & \frac{1}{2} \int_\Omega \varrho v^2(\tau) dx + \int_0^\tau \int_\Omega \mu \nabla v : \nabla v dx dt - \sum_{1 \leq l, m \leq 3} \int_0^\tau \int_\Omega \bar{B}_l \bar{B}_m \partial_{lm}^2 \eta \cdot v dx dt \\ &= \int_{\mathbb{R}^2} (g[\varrho] \eta_{+3} v_3 + \bar{B}_3 \bar{B}_l (\partial_l \eta_+ - \partial_l \eta_-) \cdot v - \kappa \sum_{i=1}^2 \partial_i \eta_{+3} \partial_i v_3) dx'. \end{aligned} \quad (4.19)$$

Now we continue to transform the above energy equality by replacing  $\eta$  with  $v$ . Since  $\eta \in C^0([0, T], L^2(\Omega))$  and  $\eta(0) = 0$ , the equation (4.7) gives

$$\eta(t, x) = \int_0^t v(s, x) ds \text{ for any } t \geq 0, \quad (4.20)$$

which, combined with (4.13), yields

$$\partial_i \eta(t, x) := \partial_i \eta_+(t, x) = \partial_i \eta_-(t, x) = \int_0^t \partial_i v(s, x) ds, \quad i = 1 \text{ or } 2. \quad (4.21)$$

Using (4.20), (4.21), Lemma 4.1 and the regularity of  $(\eta, v)$ , we deduce that

$$\begin{aligned} & \sum_{1 \leq l, m \leq 3} \int_0^\tau \int_\Omega \partial_{lm}^2 \eta \cdot v dx dt = - \sum_{1 \leq l, m \leq 3} \int_0^\tau \int_\Omega \partial_m \eta \cdot \partial_l v dx dt - \int_0^\tau \int_{\mathbb{R}^2} \llbracket \partial_3 \eta \rrbracket \cdot v dx' dt \\ &= - \sum_{1 \leq l, m \leq 3} \int_0^\tau \int_\Omega \int_0^t \partial_m v(s, x) ds \cdot \partial_l v(t, x) dx dt \\ &\quad - \int_0^\tau \int_{\mathbb{R}^2} \llbracket \partial_3 \eta \rrbracket \cdot v dx' dt, \end{aligned} \quad (4.22)$$

and

$$\sum_{i=1}^2 \int_0^\tau \int_{\mathbb{R}^2} \partial_i \eta_{+3} \partial_i v_3 dx' dt = \sum_{i=1}^2 \int_0^\tau \int_{\mathbb{R}^2} \int_0^t \partial_i v_3(s, x', 0) ds \partial_i v_3(t, x', 0) dx' dt. \quad (4.23)$$

Consequently, inserting (4.21)–(4.23) into (4.19), we arrive at

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho v^2(\tau) dx + \int_0^\tau \int_{\Omega} \mu \nabla v : \nabla v dx dt \\ & \quad + \sum_{1 \leq l, m \leq 3} \bar{B}_l \bar{B}_m \int_0^\tau \int_{\Omega} \int_0^t \partial_m v(s, x) ds \cdot \partial_l v(t, x) dx dt \\ & = g[\varrho] \int_0^\tau \int_{\mathbb{R}^2} \int_0^t v_3(s, x', 0) ds v_3(t, x', 0) dx' dt \\ & \quad - \kappa \sum_{i=1}^2 \int_0^\tau \int_{\mathbb{R}^2} \int_0^t \partial_i v_3(s, x', 0) ds \partial_i v_3(t, x', 0) dx' dt \end{aligned} \quad (4.24)$$

(2) Secondly, we further simplify the energy equality to an inequality. With the help of the regularity of  $\partial_i v_3$ , the property of absolutely continuous functions and the Fubini theorem, we conclude that

$$\begin{aligned} \int_0^\tau \int_{\mathbb{R}^2} \int_0^t \partial_i v_3(s, x', 0) ds \partial_i v_3(t, x', 0) dx' dt & = \int_{\mathbb{R}^2} \int_0^\tau \int_0^t \partial_i v_3(s, x', 0) ds \partial_i v_3(t, x', 0) dt dx' \\ & = \int_{\mathbb{R}^2} \int_0^\tau \frac{d}{dt} \left[ \int_0^t \partial_i v_3(s, x', 0) ds \right]^2 dt dx' \\ & = \int_{\mathbb{R}^2} \left[ \int_0^\tau \partial_i v_3(t, x', 0) dt \right]^2 dx' \geq 0. \end{aligned} \quad (4.25)$$

Hence, by (4.24)–(4.25), we find that

$$\begin{aligned} & \int_{\Omega} \varrho v^2(\tau) dx + 2 \sum_{1 \leq i, j \leq 3} \int_0^\tau \int_{\Omega} \mu |\partial_j v_i|^2 dx dt \\ & \leq 2g[\varrho] \int_0^\tau \int_{\mathbb{R}^2} \int_0^t v_3(s, x', 0) ds v_3(t, x', 0) dx' dt \\ & \quad - 2 \sum_{1 \leq l, m \leq 3} \bar{B}_l \bar{B}_m \int_0^\tau \int_{\Omega} \int_0^t \partial_m v(s, x) ds \cdot \partial_l v(t, x) dx' dt. \end{aligned} \quad (4.26)$$

(3) Thirdly, we start to deduce the local-in-time uniqueness from the inequality (4.26). Analogously to (4.25), the first integral on the right-hand side of (4.26) can be bounded as follows.

$$\begin{aligned} & 2 \int_0^\tau \int_{\mathbb{R}^2} \int_0^t v_3(s, x', 0) v_3(t, x', 0) dx' ds dt = \int_{\mathbb{R}^2} \left( \int_0^\tau v_3(t, x', 0) dt \right)^2 dx' \\ & \leq \tau \int_0^\tau \int_{\mathbb{R}^2} |v_3(t, x', 0)|^2 dx' dt \\ & = 2\tau \int_0^\tau \int_{\mathbb{R}^2} \int_1^0 u_3(t, x', x_3) \partial_3 v_3(t, x', x_3) dx_3 dx' dt \\ & \leq \tau \int_0^\tau \int_{\mathbb{R}^2} \left( \frac{\mu}{\tau g[\varrho]} \int_0^1 |\partial_3 v_3(t)|^2 dx_3 + \frac{\tau g[\varrho]}{\mu} \int_0^1 |v_3(t)|^2 dx_3 \right) dx' dt \\ & \leq \frac{1}{g[\varrho]} \int_0^\tau \|\sqrt{\mu} \partial_3 v_3(t)\|_{L^2(\Omega)}^2 dt + \frac{\tau^2 g[\varrho]}{\mu_-} \int_0^\tau \|v(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (4.27)$$

On the other hand, letting  $B_0 = \max\{|\bar{B}_1|, |\bar{B}_2|, |\bar{B}_3|, 1\}$ , and using the Hölder, Minkowski and Cauchy-Schwarz inequalities, the second integral on the right-hand side of (4.26) can be estimates as follows.

$$\begin{aligned}
& \sum_{1 \leq l, m \leq 3} \bar{B}_l \bar{B}_m \int_0^\tau \int_\Omega \int_0^t \partial_m v(s, x) ds \cdot \partial_l v(t, x) dx dt \\
& \leq B_0^2 \sum_{1 \leq l, m \leq 3} \sum_{i=1}^3 \int_0^\tau \left[ \int_\Omega \left( \int_0^t \partial_m v_i(s, x) ds \right)^2 dx \right]^{1/2} \left( \int_\Omega |\partial_l v_i(t, x)|^2 dx \right)^{1/2} dt \\
& \leq B_0^2 \sum_{1 \leq l, m \leq 3} \sum_{i=1}^3 \int_0^\tau \left[ \int_0^t \left( \int_\Omega |\partial_m v_i(s, x)|^2 dx \right)^{1/2} ds \right] \left( \int_\Omega |\partial_l v_i(t, x)|^2 dx \right)^{1/2} dt \quad (4.28) \\
& \leq B_0^2 \sum_{1 \leq l, m \leq 3} \sum_{i=1}^3 \left[ \int_0^\tau \left( \int_\Omega |\partial_m v_i(s, x)|^2 dx \right)^{1/2} ds \right] \left[ \int_0^\tau \left( \int_\Omega |\partial_l v_i(t, x)|^2 dx \right)^{1/2} dt \right] \\
& \leq \tau B_0^2 \sum_{1 \leq i, j \leq 3} \int_0^\tau \int_\Omega |\partial_j v_i(t, x)|^2 dx dt.
\end{aligned}$$

Substituting (4.27)–(4.28) into (4.26), we deduce that

$$\begin{aligned}
& \|\sqrt{\varrho}v(\tau)\|_{L^2(\Omega)}^2 + \mu_- \sum_{1 \leq i, j \leq 3} \int_0^\tau \|\partial_j v_i\|_{L^2(\Omega)}^2 dt \\
& \leq \tau^2 g^2 [\varrho]^2 \mu_-^{-1} \int_0^\tau \|v(t)\|_{L^2(\Omega)}^2 dt + \tau B_0^2 \sum_{1 \leq i, j \leq 3} \int_0^\tau \|\partial_j v_i(t, x)\|_{L^2(\Omega)}^2 dt. \quad (4.29)
\end{aligned}$$

Now, taking  $t_1 = \min\left\{\frac{\mu_1}{2B_0^2}, T\right\}$ , the inequality (4.29) implies

$$\|v(\tau)\|_{L^2(\Omega)}^2 \leq \frac{t_1^2 g^2 [\varrho]^2}{\mu_- \varrho_-} \int_0^\tau \|v(t)\|_{L^2(\Omega)}^2 dt, \quad \tau \in (0, t_1]. \quad (4.30)$$

Moreover, if we apply the Grownwall inequality to (4.30), we see that

$$\|v(\tau)\|_{L^2(\Omega)}^2 = 0 \quad \text{for any } \tau \in [0, t_1],$$

which yields  $v = 0$ , i.e.,  $\tilde{v} = \bar{v}$ . This, combined with (4.8) and (4.20), proves that

$$(\tilde{\eta}, \tilde{v}, \nabla \tilde{q}) = (\bar{\eta}, \bar{v}, \nabla \bar{q}) \quad \text{for any } \tau \in (0, t_1]. \quad (4.31)$$

(4) Finally, the local-in-time uniqueness of the solution  $(\tilde{\eta}, \tilde{v})$  can be continued onto  $[0, T]$ . In fact, if  $t_1 < T$ , we can take  $(\tilde{\eta}, \tilde{v})(t_1) = (\bar{\eta}, \bar{v})(t_1)$  as the initial data, and continue the above procedure (1)–(3) to obtain the uniqueness of solutions  $(\tilde{\eta}, \tilde{v})$  on  $(0, t_2]$ , where  $t_2 = 2t_1 = \mu_1^2/(2b_0^2)$  or  $t_2 = T$ . Hence, after repeating this procedure of extending time interval by finite times, (4.31) holds for any  $\tau \in (0, T]$ .  $\square$

#### 4.2. Proof of Theorem 2.1

We define

$$\beta_1 = d_1 + (d_2 - d_1)/3, \quad \beta_2 = d_1 + 2(d_2 - d_1)/3.$$

Fix  $j \geq k \geq 0$ ,  $\alpha > 0$  and let  $\tilde{c}_j$  be the constants from Theorem 3.3. For each  $n \in \mathbb{N}$ , let  $t_n$  satisfy

$$e^{2t_n\lambda_0} = \alpha^2 n^2 \tilde{c}_j^2, \quad (4.32)$$

i.e.,

$$t_n = \frac{\ln \tilde{c}_j}{\lambda_0} + \frac{\ln(\alpha n)}{\lambda_0} := C_{jk} + C_1 \ln(\alpha n),$$

where  $\lambda_0$  is defined by (3.42). Choose  $f_n \in C_0^\infty(\mathbb{R}^2)$ , such that  $\text{supp}(f_n) \subset B(0, \beta_2) \setminus B(0, \beta_1)$ , where  $f_n$  is real-valued and radial, and

$$\int_{\mathbb{R}^2} f_n^2(|\xi|) d\xi = \frac{1}{\tilde{c}_j^2 n^2}. \quad (4.33)$$

Now, we can apply Theorem 3.3 with  $f = f_n$  to find that  $(\eta_n(t), v_n(t), q_n(t)) \in \bar{H}^j(\Omega)$  solves the problem (1.20)–(1.22). It follows thus from (3.68) and (4.33) that (2.2) holds for all  $n$ .

Recalling  $\text{supp}(f_n) \subset B(\beta_1, \beta_2)$  and  $\lambda(|\xi_1|, |\xi_2|) \geq \lambda_0$ , we have, after a straightforward calculation and using (3.66), (4.33), (4.32) and (3.53), that

$$\begin{aligned} \|v_n(t)\|_{\bar{H}^k}^2 &\geq \int_{\mathbb{R}^2} (1 + |\xi|^2)^k e^{2t\lambda(|\xi_1|, |\xi_2|)} f_n^2(|\xi|) \|w(\xi, x_3)\|_{L^2(-1,1)}^2 d\xi \\ &\geq e^{2t\lambda_0} \int_{\mathbb{R}^2} f_n^2(|\xi|) \|w(\xi, x_3)\|_{L^2(-1,1)}^2 d\xi \\ &= e^{2(t-t_n)\lambda_0} \alpha^2 n^2 \tilde{c}_j^2 \int_{\mathbb{R}^2} f_n^2(|\xi|) d\xi \\ &\geq \alpha^2 \quad \text{for any } t \geq t_n, \end{aligned}$$

which, together with  $\eta_n(t, x) = \lambda(|\xi_1|, |\xi_2|) v_n(t, x)$ , implies (2.3) and (2.4). This completes the proof of Theorem 2.1

## 5. Proof of Theorem 2.2

In this section we show Theorem 2.2. The main idea of our proof comes from [6, 10] but with more complicated computations. We argue by contradiction. Therefore, we suppose that the perturbed problem has the global stability of order  $k$  for some  $k \geq 3$ .

Let  $\delta, C_2 > 0$  and  $F : [0, \delta] \rightarrow \mathbb{R}^+$  be the constants and function provided by the global stability of order  $k$ , respectively. Fixing  $n \in \mathbb{N}$  such that  $n > C_2$ . Applying Theorem 2.1 with this  $n$ ,  $t_n = T/2$ ,  $k \geq 3$ , and  $\alpha = 2$ , we find that  $(\tilde{\eta}, \tilde{v}, \tilde{\sigma})$  solves (1.20), satisfying

$$\|\tilde{\eta}(0)\|_{\bar{H}^k} + \|\tilde{v}(0)\|_{\bar{H}^k} < n^{-1}$$

but

$$\|\tilde{v}(t)\|_{\bar{H}^3} \geq 2 \quad \text{for } t \geq T/2. \quad (5.1)$$

For  $\varepsilon > 0$  we define  $\bar{\eta}_0^\varepsilon = \varepsilon \tilde{\eta}(0)$  and  $\bar{v}_0^\varepsilon = \varepsilon \tilde{v}(0)$ . Then, for  $\varepsilon < \delta n$ ,  $\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon)\|_{\bar{H}^k} < \delta$ . So, according to the global stability of order  $k$ , there exist  $\eta^\varepsilon, v^\varepsilon, q^\varepsilon$  that solve the perturbed problem (2.5)–(2.7) with  $\bar{B} = (B, 0, 0)$ , i.e.,

$$\begin{cases} \partial_t \eta^\varepsilon = v^\varepsilon, \\ \varrho \partial_t v_i^\varepsilon + (I_{jk} - G_{jk}^\varepsilon) \partial_k T_{ij}^\varepsilon = B^2 \partial_{11}^2 \eta_i^\varepsilon, \quad i = 1, 2, 3, \\ \text{div} v^\varepsilon - \text{tr}(G^\varepsilon Dv^\varepsilon) = 0 \end{cases} \quad (5.2)$$

with jump conditions across the interface

$$[[v^\varepsilon]] = 0, \quad [[T_{ij}^\varepsilon n_j^\varepsilon]] = g[\varrho]\eta_3^\varepsilon n_i^\varepsilon + B^2(e_1 + \partial_1 \eta^\varepsilon) \cdot n^\varepsilon [[\partial_1 \eta_i^\varepsilon]] + \kappa H^\varepsilon n_i^\varepsilon \quad (5.3)$$

and initial data satisfying  $\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon)\|_{\bar{H}^k} < \delta$ , where

$$\begin{aligned} (G^\varepsilon)^T &:= (G_{jk}^\varepsilon)_{3 \times 3}^T := I - (I + D\eta^\varepsilon)^{-1}, \\ T_{ij}^\varepsilon &= q^\varepsilon I_{ij} - \mu((I_{jk} - G_{jk}^\varepsilon)\partial_k v_i^\varepsilon + (I_{ik} - G_{ik}^\varepsilon)\partial_k v_j^\varepsilon), \\ n^\varepsilon &= N^\varepsilon/|N^\varepsilon| \text{ with } N^\varepsilon = (e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \times (e_2 + \varepsilon \partial_2 \bar{\eta}^\varepsilon) \\ &= e_3 + \varepsilon(e_1 \times \partial_2 \bar{\eta}^\varepsilon + \partial_1 \bar{\eta}^\varepsilon \times e_2) + \varepsilon^2(\partial_1 \bar{\eta}^\varepsilon \times \partial_2 \bar{\eta}^\varepsilon) \\ &=: e_3 + \varepsilon \bar{N}^\varepsilon, \end{aligned} \quad (5.4)$$

and

$$H^\varepsilon = \left( \frac{|e_1 + \partial_1 \eta^\varepsilon|^2 \partial_2^2 \eta^\varepsilon - 2(e_1 + \partial_1 \eta^\varepsilon) \cdot (e_2 + \partial_2 \eta^\varepsilon) \partial_1 \partial_2 \eta^\varepsilon + |e_2 + \partial_2 \eta^\varepsilon|^2 \partial_1^2 \eta^\varepsilon}{|e_1 + \partial_1 \eta^\varepsilon|^2 |e_2 + \partial_2 \eta^\varepsilon|^2 - |(e_1 + \partial_1 \eta^\varepsilon) \cdot (e_2 + \partial_2 \eta^\varepsilon)|^2} \right) \cdot n^\varepsilon. \quad (5.5)$$

Here we have used the Einstein convention of summing over repeated indices, and  $\eta^\varepsilon$  to denote the both case  $\eta_-^\varepsilon$  and  $\eta_+^\varepsilon$  at the interface  $\{x_3 = 0\}$  in (5.3), (5.4) and (5.5), except for the notation  $[[\partial_i \eta_i^\varepsilon]]$  (see Remark 2.1). Moreover,  $\eta^\varepsilon, v^\varepsilon, q^\varepsilon$  satisfy

$$\sup_{0 \leq t < T} \|(\eta^\varepsilon, v^\varepsilon, q^\varepsilon)(t)\|_{\bar{H}^3} \leq F(\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon)\|_{\bar{H}^3}) \leq C_2 \varepsilon \|(\tilde{\eta}^\varepsilon, \tilde{v}^\varepsilon)(0)\|_{\bar{H}^3} < \varepsilon, \quad (5.6)$$

and  $\eta_\pm^\varepsilon \in C^2(\bar{\Omega}_\pm)$  when  $\kappa > 0$ .

Now, defining the rescaled functions  $\bar{\eta}^\varepsilon = \eta^\varepsilon/\varepsilon$ ,  $\bar{v}^\varepsilon = v^\varepsilon/\varepsilon$ ,  $\bar{q}^\varepsilon = q^\varepsilon/\varepsilon$ , and rescaling (5.6), one gets

$$\sup_{0 \leq t < T} \|(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{q}^\varepsilon)(t)\|_{\bar{H}^3} \leq 1, \quad (5.7)$$

which implies that

$$\sup_{0 \leq t < T} \|(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{q}^\varepsilon)(t)\|_{C^1(\bar{\Omega}_+)} + \sup_{0 \leq t < T} \|(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{q}^\varepsilon)(t)\|_{C^1(\bar{\Omega}_-)} \leq K_1, \quad (5.8)$$

for some constant  $K_1$ . On the other hand, using the jump conditions (5.3) in term of  $v^\varepsilon$ , one has

$$\sup_{0 \leq t < T} \|\bar{v}^\varepsilon(t)\|_{H_0^1(\Omega)} \leq 1. \quad (5.9)$$

Note that by construction,  $(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon)(0) = (\tilde{\eta}, \tilde{v})(0)$ . Our next goal is to show that the rescaled functions converge as  $\varepsilon \rightarrow 0$  to the solution  $(\bar{\eta}, \bar{v}, \bar{\sigma})$  of the linearized equations (1.20) with  $\bar{B} = (B, 0, 0)$  and the corresponding linearized jump conditions.

### 5.1. Convergence to the linearized equations

We may further assume that  $\varepsilon$  is sufficiently small, so that

$$\sup_{0 \leq t < T} \|\varepsilon D\bar{\eta}^\varepsilon(t)\|_{L^\infty(\Omega)} < 1/9 \quad \text{and} \quad \varepsilon < 1/(2K_2), \quad (5.10)$$

where  $K_2 > 0$  is the best constant in the inequality

$$\|FG\|_{\bar{H}^2} \leq K_2 \|F\|_{\bar{H}^2} \|G\|_{\bar{H}^2}$$

for  $3 \times 3$  matrix-valued functions  $F, G$ . Then,

$$\bar{G}^\varepsilon = (I - (I + \varepsilon(D\bar{\eta}^\varepsilon)^T)^{-1})/\varepsilon$$

is well-defined by (5.10) and the uniform boundedness in  $L^\infty(0, \infty; \bar{H}^2(\Omega))$ , since

$$\begin{aligned} \|\bar{G}^\varepsilon\|_{\bar{H}^2} &= \left\| \sum_{n=1}^{\infty} (-\varepsilon)^{n-1} (D\bar{\eta}^\varepsilon)^n \right\|_{\bar{H}^2} \leq \sum_{n=1}^{\infty} \varepsilon^{n-1} \|(D\bar{\eta}^\varepsilon)^n\|_{\bar{H}^2} \\ &\leq \sum_{n=1}^{\infty} (\varepsilon K_1)^{n-1} \|D\bar{\eta}^\varepsilon\|_{\bar{H}^2}^n \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \|\bar{\eta}^\varepsilon\|_{\bar{H}^3}^n < \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2. \end{aligned} \quad (5.11)$$

Next, we exploit the boundedness of  $\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon$  and  $\bar{G}^\varepsilon$  to control  $\partial_t \bar{\eta}^\varepsilon, \partial_t \bar{v}^\varepsilon$  and to give some convergence results. The first equation in (5.2) implies that  $\partial_t \bar{\eta}^\varepsilon = \bar{v}^\varepsilon$ , therefore

$$\sup_{0 \leq t < T} \|\partial_t \bar{\eta}^\varepsilon(t)\|_{\bar{H}^3} = \sup_{0 \leq t < T} \|\bar{v}^\varepsilon(t)\|_{\bar{H}^3} \leq 1. \quad (5.12)$$

By virtue of (5.7) and (5.11), the third equation in (5.2) yields

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t < T} \|\operatorname{div} \bar{v}^\varepsilon(t)\|_{\bar{H}^2} = \lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t < T} \|\varepsilon \operatorname{tr}(\bar{G}^\varepsilon D\bar{v}^\varepsilon)(t)\|_{\bar{H}^2} = 0. \quad (5.13)$$

Expanding the second equation in (5.2), one sees that

$$\begin{aligned} &\varrho \partial_t \bar{v}_i^\varepsilon + \partial_i \bar{q}^\varepsilon - \mu \Delta \bar{v}_i^\varepsilon - B^2 \partial_{11}^2 \bar{\eta}_i^\varepsilon \\ &= \varepsilon \bar{G}_{jk}^\varepsilon (\partial_k (\bar{q}^\varepsilon I_{ij} - \mu ((I_{jk} - \varepsilon \bar{G}_{jk}^\varepsilon) \partial_k \bar{v}_i^\varepsilon + (I_{ik} - \varepsilon \bar{G}_{ik}^\varepsilon) \partial_k \bar{v}_j^\varepsilon)) \\ &\quad - \varepsilon \mu \partial_j (\bar{G}_{jj}^\varepsilon \partial_j \bar{v}_i^\varepsilon + \bar{G}_{ij}^\varepsilon \partial_j \bar{v}_j^\varepsilon) := \bar{X}^\varepsilon, \quad i = 1, 2, 3, \end{aligned} \quad (5.14)$$

whence, by (5.7) and (5.11),

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t < T} \|\bar{X}^\varepsilon\|_{\bar{H}^1} = 0 \quad (5.15)$$

and

$$\sup_{0 \leq t < T} \|\partial_t \bar{v}^\varepsilon\|_{\bar{H}^1} \leq K_3 \quad \text{for some constant } K_3 > 0. \quad (5.16)$$

By (5.7), (5.9), (5.16) and the sequential weak-\*compactness, we see that up to extraction of a subsequence (which we still denote using only  $\varepsilon$ ),

$$(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{q}^\varepsilon) \rightarrow (\bar{\eta}, \bar{v}, \bar{q}) \text{ weakly-* in } L^\infty(0, T; \bar{H}^3(\Omega)), \quad (5.17)$$

$$\bar{v}^\varepsilon \rightarrow \bar{v} \text{ weakly-* in } L^\infty(0, T; H_0^1(\Omega))$$

and

$$\partial_t \bar{v}^\varepsilon \rightarrow \partial_t \bar{v} \text{ weakly-* in } L^\infty(0, T; \bar{H}^1(\Omega)). \quad (5.18)$$

From the lower semi-continuity one gets

$$\operatorname{ess\,sup}_{0 \leq t < T} \|(\bar{\eta}, \bar{v}, \bar{q})(t)\|_{\bar{H}^3} \leq 1, \quad \operatorname{ess\,sup}_{0 \leq t < T} \|\bar{v}\|_{H_0^1(\Omega)} \leq 1. \quad (5.19)$$

On the other hand, if we use (5.12), (5.16), and (5.7), an abstract version of the Arzela-Ascoli theorem (see [12, Theorem 1.70]), and the Aubin-Lions Theorem (see [12, Theorem 1.71]), we obtain that

$$(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon) \rightarrow (\bar{\eta}, \bar{v}) \text{ strongly in } L^p(0, T; \bar{H}^2(\Omega)) \text{ and } C^0([0, T], L^2(\Omega)) \quad (5.20)$$

for any  $1 \leq p < \infty$ . This strong convergence, together with the equation  $\partial_t \bar{\eta}^\varepsilon = \bar{v}^\varepsilon$ , gives that

$$\partial_t \bar{\eta}^\varepsilon \rightarrow \partial_t \bar{\eta} \quad \text{strongly in } L^p(0, T; \bar{H}^2(\Omega)). \quad (5.21)$$

Rescaling the equations (5.2), utilizing (5.13)–(5.15), (5.17), (5.18) and (5.21), the resulting equations thus imply that

$$\begin{cases} \partial_t \bar{\eta} = \bar{v}, \\ \varrho \partial_t \bar{v} + \nabla \bar{\sigma} = \mu \Delta \bar{v} + B^2 \partial_{11}^2 \bar{\eta}, \\ \operatorname{div} \bar{v} = 0 \end{cases} \quad (5.22)$$

holds a.e. in  $(0, T) \times \Omega$ . From (5.20) and the initial conditions  $(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon)(0) = (\tilde{\eta}, \tilde{v})(0)$ , it follows that

$$(\bar{\eta}, \bar{v})(0) = (\tilde{\eta}, \tilde{v})(0) \quad (5.23)$$

as well.

### 5.2. Convergence to the linearized jump conditions

We proceed to show some convergence results for the jump conditions. Multiplying (5.2)<sub>2</sub> with  $\varphi \in (C_0^\infty((0, T) \times \Omega))^3$ , rescaling the resulting equation, we find that

$$\int_0^T \int_\Omega (\varrho \partial_t \bar{v}^\varepsilon - B^2 \partial_{11}^2 \bar{\eta}^\varepsilon) \cdot \varphi \, dx \, dt + \int_0^T \int_\Omega (I_{jk} - \varepsilon \bar{G}_{jk}^\varepsilon) \varphi_i \partial_k \bar{T}_{ij}^\varepsilon \, dx \, dt = 0. \quad (5.24)$$

Integrating by parts and making use of the second jump condition in (5.3), we conclude that

$$\begin{aligned} & \int_0^T \int_\Omega \varphi_i \partial_j \bar{T}_{ij}^\varepsilon \, dx \, dt + \int_0^T \int_\Omega \bar{T}_{ij}^\varepsilon \partial_j \varphi_i \, dx \, dt = - \int_0^T \int_{\mathbb{R}^2} \varphi_i \llbracket \bar{T}_{i3}^\varepsilon \rrbracket \, dx' \, dt \\ & = - \int_0^T \int_{\mathbb{R}^2} \varphi_i \llbracket \bar{T}_{ij}^\varepsilon n_j \rrbracket \, dx' \, dt + \int_0^T \int_{\mathbb{R}^2} \varphi_i (\llbracket \bar{T}_{ij}^\varepsilon n_j \rrbracket - \llbracket \bar{T}_{i3}^\varepsilon \rrbracket) \, dx' \, dt \\ & = - \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3^\varepsilon n^\varepsilon + B^2 (e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \cdot n^\varepsilon \llbracket \partial_1 \bar{\eta}^\varepsilon \rrbracket + \kappa \bar{H}^\varepsilon n^\varepsilon) \cdot \varphi \, dx' \, dt \\ & \quad + \int_0^T \int_{\mathbb{R}^2} \varphi_i (\llbracket \bar{T}_{ij}^\varepsilon n_j \rrbracket - \llbracket \bar{T}_{i3}^\varepsilon \rrbracket) \, dx' \, dt, \end{aligned} \quad (5.25)$$

where  $\bar{H}^\varepsilon = H^\varepsilon / \varepsilon$ . On the other hand,

$$\begin{aligned} \int_0^T \int_\Omega \bar{T}_{ij}^\varepsilon \partial_j \varphi_i \, dx \, dt &= \int_0^T \int_\Omega (\bar{q}^\varepsilon \operatorname{div} \varphi - \mu (\nabla \bar{v}^\varepsilon + \nabla (\bar{v}^\varepsilon)^T) : \nabla \varphi) \, dx \, dt \\ &\quad + \varepsilon \int_0^T \int_\Omega \mu (\bar{G}_{jk}^\varepsilon \partial_k \bar{v}_i^\varepsilon + \bar{G}_{ik}^\varepsilon \partial_k \bar{v}_j^\varepsilon) \partial_j \varphi_i \, dx \, dt \end{aligned} \quad (5.26)$$

Putting (5.24)–(5.26) together, we arrive at

$$\begin{aligned} & \int_0^T \int_\Omega (\varrho \partial_t \bar{v}^\varepsilon - B^2 \partial_{11}^2 \bar{\eta}^\varepsilon) \cdot \varphi \, dx \, dt + \int_0^T \int_\Omega (\mu (\nabla \bar{v}^\varepsilon + \nabla (\bar{v}^\varepsilon)^T) : \nabla \varphi - \bar{q}^\varepsilon \operatorname{div} \varphi) \, dx \, dt \\ & \quad - \varepsilon \left[ \int_0^T \int_\Omega \varphi_i \bar{G}_{jk}^\varepsilon \partial_k \bar{T}_{ij}^\varepsilon \, dx \, dt + \int_0^T \int_\Omega \mu (\bar{G}_{jk}^\varepsilon \partial_k \bar{v}_i^\varepsilon + \bar{G}_{ik}^\varepsilon \partial_k \bar{v}_j^\varepsilon) \partial_j \varphi_i \, dx \, dt \right] \\ & = \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3^\varepsilon n^\varepsilon + B^2 (e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \cdot n^\varepsilon \llbracket \partial_1 \bar{\eta}^\varepsilon \rrbracket + \kappa \bar{H}^\varepsilon n^\varepsilon) \cdot \varphi \, dx' \, dt \\ & \quad - \int_0^T \int_{\mathbb{R}^2} \varphi_i (\llbracket \bar{T}_{ij}^\varepsilon n_j \rrbracket - \llbracket \bar{T}_{i3}^\varepsilon \rrbracket) \, dx' \, dt. \end{aligned} \quad (5.27)$$



Next, we deal with the limit of the equality (5.27) as  $\varepsilon \rightarrow 0$ . Obviously, by (5.17), (5.18), the bounds (5.7) and (5.11), it is easy to verify that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (\varrho \partial_t \bar{v}^\varepsilon - B^2 \partial_{11}^2 \bar{\eta}^\varepsilon) \cdot \varphi dx dt \\ & + \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (\mu(\nabla \bar{v}^\varepsilon + \nabla(\bar{v}^\varepsilon)^T) : \nabla \varphi - \bar{q}^\varepsilon \operatorname{div} \varphi) dx dt \\ & = \int_0^T \int_{\Omega} (\varrho \partial_t \bar{v} - B^2 \partial_{11}^2 \bar{\eta}) \cdot \varphi dx dt + \int_0^T \int_{\Omega} (\mu(\nabla \bar{v} + \nabla \bar{v}^T) : \nabla \varphi - \bar{q} \operatorname{div} \varphi) dx dt, \end{aligned} \quad (5.28)$$

and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \left[ \int_0^T \int_{\Omega} \varphi_i \bar{G}_{jk}^\varepsilon \partial_k \bar{T}_{ij}^\varepsilon dx dt + \int_0^T \int_{\Omega} \mu(\bar{G}_{jk}^\varepsilon \partial_k \bar{v}_i^\varepsilon + \bar{G}_{ik}^\varepsilon \partial_k \bar{v}_j^\varepsilon) \partial_j \varphi_i dx dt \right] = 0. \quad (5.29)$$

Thus, it remains to analyze the convergence of the two integrals on the right hand of (5.27). To this end, using (5.4) and (5.5), we rewrite the first integral as

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3^\varepsilon n^\varepsilon + B^2(e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \cdot n^\varepsilon [\partial_1 \bar{\eta}^\varepsilon] + \kappa \bar{H}^\varepsilon n^\varepsilon) \cdot \varphi dx' dt \\ & = \int_0^T \int_{\mathbb{R}^2} \left\{ \frac{g[\varrho] \bar{\eta}_3^\varepsilon e_3}{|e_3 + \varepsilon \bar{N}^\varepsilon|} + \frac{\kappa \Delta_{x'} \bar{\eta}_3^\varepsilon e_3}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} \right. \\ & \quad \left. + \varepsilon \left[ \frac{B^2 \bar{F}^\varepsilon + g[\varrho] \bar{\eta}_3^\varepsilon \bar{N}^\varepsilon}{|e_3 + \varepsilon \bar{N}^\varepsilon|} + \frac{\kappa \bar{L}^\varepsilon}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} \right] \right\} \cdot \varphi dx' dt, \end{aligned} \quad (5.30)$$

where

$$\begin{aligned} \bar{R}^\varepsilon &= 2 \sum_{i=1}^2 \partial_i \bar{\eta}_i^\varepsilon + \varepsilon \sum_{i=1}^2 |\partial_i \bar{\eta}^\varepsilon|^2 + \varepsilon (2 \partial_1 \bar{\eta}_1^\varepsilon + \varepsilon |\partial_1 \bar{\eta}^\varepsilon|^2) (2 \partial_2 \bar{\eta}_2^\varepsilon + \varepsilon |\partial_2 \bar{\eta}^\varepsilon|^2) \\ & \quad - \varepsilon (\partial_1 \bar{\eta}_2^\varepsilon + \partial_2 \bar{\eta}_1^\varepsilon + \varepsilon \partial_1 \bar{\eta}^\varepsilon \cdot \partial_2 \bar{\eta}^\varepsilon)^2, \\ \bar{F}^\varepsilon &= (\bar{N}_1^\varepsilon + \partial_1 \bar{\eta}_3^\varepsilon + \varepsilon \partial_1 \bar{\eta}^\varepsilon \cdot \bar{N}^\varepsilon) [\partial_1 \bar{\eta}^\varepsilon], \\ \bar{L}^\varepsilon &= \{(2 \partial_1 \bar{\eta}_1^\varepsilon + \varepsilon |\partial_1 \bar{\eta}^\varepsilon|^2) \partial_2^2 \bar{\eta}_3^\varepsilon + (2 \partial_2 \bar{\eta}_2^\varepsilon + \varepsilon |\partial_2 \bar{\eta}^\varepsilon|^2) \partial_1^2 \bar{\eta}_3^\varepsilon \\ & \quad - 2(\partial_1 \bar{\eta}_2^\varepsilon + \partial_2 \bar{\eta}_1^\varepsilon + \varepsilon \partial_1 \bar{\eta}^\varepsilon \cdot \partial_2 \bar{\eta}^\varepsilon) \partial_1 \partial_2 \bar{\eta}_3^\varepsilon \\ & \quad + \bar{\mathcal{H}}^\varepsilon \cdot \bar{N}^\varepsilon\} (e_3 + \bar{N}^\varepsilon) + \Delta_{x'} \bar{\eta}_3^\varepsilon \bar{N}^\varepsilon, \end{aligned}$$

and

$$\bar{\mathcal{H}}^\varepsilon = |e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon|^2 \partial_2^2 \bar{\eta}^\varepsilon - 2(e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \cdot (e_2 + \varepsilon \partial_2 \bar{\eta}^\varepsilon) \partial_1 \partial_2 \bar{\eta}^\varepsilon + |e_2 + \varepsilon \partial_2 \bar{\eta}^\varepsilon|^2 \partial_1^2 \bar{\eta}^\varepsilon.$$

Clearly, by the trace theorem, (5.7) and (5.8) in terms of  $\bar{\eta}^\varepsilon$ ,  $\sup_{0 \leq t < T} \|(\bar{R}^\varepsilon, \bar{F}^\varepsilon, \bar{N}^\varepsilon)(t)\|_{L^\infty(\mathbb{R}^2)}$  and  $\sup_{0 \leq t < T} \|\bar{L}^\varepsilon(t)\|_{L^2(\mathbb{R}^2)}$  are uniformly bounded. Hence, as  $\varepsilon \rightarrow 0$ , we have  $1 + \varepsilon \bar{R}^\varepsilon > 0$ ,  $|e_3 + \varepsilon \bar{N}^\varepsilon| > 0$ , and

$$\varepsilon \left[ \frac{B^2 \bar{F}^\varepsilon + g[\varrho] \bar{\eta}_3^\varepsilon \bar{N}^\varepsilon}{|e_3 + \varepsilon \bar{N}^\varepsilon|} + \frac{\kappa \bar{L}^\varepsilon}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} \right] \rightarrow 0 \text{ strongly in } L^\infty(0, T; L^2(\mathbb{R}^2)). \quad (5.31)$$

Moreover, by virtue of (5.20), (5.7) and (5.8) in terms of  $\bar{\eta}^\varepsilon$ , using integrating by parts, the trace

theorem and dominated convergence theorem, we deduce that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^2} \frac{\Delta_{x'} \bar{\eta}_3^\varepsilon e_3 \cdot \varphi}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} dx' dt \\
&= - \sum_{i=1}^2 \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^2} \frac{\partial_i \bar{\eta}_3^\varepsilon \partial_i \varphi_3}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} dx' dt \\
&\quad - \sum_{i=1}^2 \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^2} \varphi_i \partial_i \bar{\eta}_3^\varepsilon \partial_i \left[ \frac{1}{(1 + \varepsilon \bar{R}^\varepsilon) |e_3 + \varepsilon \bar{N}^\varepsilon|^2} \right] dx' dt \\
&= - \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^2} \partial_i \bar{\eta}_3 \partial_i \varphi_3 dx' dt
\end{aligned} \tag{5.32}$$

and

$$\int_0^T \int_{\mathbb{R}^2} \frac{g[\varrho] \bar{\eta}_3^\varepsilon e_3 \cdot \varphi}{|e_3 + \varepsilon \bar{N}^\varepsilon|} dx' dt \rightarrow \int_0^T \int_{\mathbb{R}^2} g[\varrho] \bar{\eta}_3 \varphi_3 dx' dt. \tag{5.33}$$

In view of (5.31)–(5.33) and (5.30), we find that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3^\varepsilon n^\varepsilon + B^2(e_1 + \varepsilon \partial_1 \bar{\eta}^\varepsilon) \cdot n^\varepsilon [\partial_1 \bar{\eta}_i^\varepsilon] + \kappa \bar{H}^\varepsilon n^\varepsilon) \cdot \varphi dx' dt \\
&= \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3 \varphi_3 - \kappa \sum_{i=1}^2 \partial_i \bar{\eta}_3 \partial_i \varphi_3) dx' dt.
\end{aligned} \tag{5.34}$$

Finally, using (5.8), we easily verify that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^2} \varphi_i ([\bar{T}_{ij}^\varepsilon n_j] - [\bar{T}_{i3}^\varepsilon]) dx' dt = 0. \tag{5.35}$$

Consequently, thanks to (5.28), (5.29), (5.34) and (5.35), we take to the limit in (5.27) to arrive at

$$\begin{aligned}
& \int_0^T \int_{\Omega} (\varrho \partial_t \bar{v} - B^2 \partial_{11}^2 \bar{\eta}) \varphi dx dt + \int_0^T \int_{\Omega} (\mu(\nabla \bar{v} + \nabla \bar{v}^T) : \nabla \varphi - \bar{q} \operatorname{div} \varphi) dx dt \\
&= \int_0^T \int_{\mathbb{R}^2} (g[\varrho] \bar{\eta}_3 e_3 - \kappa \sum_{i=1}^2 \partial_i \bar{\eta}_3 \partial_i \varphi_3) dx' dt.
\end{aligned} \tag{5.36}$$

On the other hand, similarly to (4.17), we multiply (5.22)<sub>2</sub> by  $\varphi$  and integrate over  $(0, T) \times \Omega$  to obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega} (\varrho \partial_t \bar{v} - B^2 \partial_{11}^2 \bar{\eta}) \cdot \varphi dx dt + \int_0^T \int_{\Omega} (\mu(\nabla \bar{v} + \nabla \bar{v}^T) : \nabla \varphi - \bar{q} \operatorname{div} \varphi) dx dt \\
&= \int_0^T \int_{\mathbb{R}^2} \left( \bar{q}_+ I - \mu_+(\nabla \bar{v}_+ + \nabla \bar{v}_+^T) - (\bar{q}_- I - \mu_-(\nabla \bar{v}_- + \nabla \bar{v}_-^T)) \right) e_3 \cdot \varphi dx' dt,
\end{aligned} \tag{5.37}$$

where we have used the fact that  $\gamma_\pm(q) = q_\pm$  at the interface  $\{x_3 = 0\}$  since  $q \in \bar{H}^3(\Omega)$  for a.e.  $t \in (0, T)$ . Comparing (5.36) with (5.37), we conclude that

$$\begin{aligned}
& \int_{\mathbb{R}^2} \left( \bar{q}_+ I - \mu_+(\nabla \bar{v}_+ + \nabla \bar{v}_+^T) - (\bar{q}_- I - \mu_-(\nabla \bar{v}_- + \nabla \bar{v}_-^T)) \right) e_3 \cdot \phi dx' \\
&= \int_{\mathbb{R}^2} \left( g[\varrho] \bar{\eta}_3 \phi_3 - \kappa \sum_{i=1}^2 \partial_i \bar{\eta}_3 \partial_i \phi_3 \right) dx' \quad \text{for a.e. } t \in (0, T)
\end{aligned} \tag{5.38}$$

holds for any  $\phi \in H_0^1(\mathbb{R}^2)$ .

### 5.3. Contradiction argument

Notice that  $\bar{\eta}_3$  in (5.38) includes both cases of  $\bar{\eta}_{+3}$  and  $\bar{\eta}_{-3}$ . In view of Definition 4.1, we find that  $(\bar{\eta}, \bar{v}, \bar{q})$  is just a strong solution of the linearized problem (1.20)–(1.22). By Remark 4.3,  $(\tilde{\eta}, \tilde{v}, \tilde{q})$  constructed in Theorem 2.1 is also a strong solution of (1.20)–(1.22). Moreover,  $\tilde{\eta}(0) = \bar{\eta}(0)$  and  $\tilde{v}(0) = \bar{v}(0)$  (see (5.23)). Then, according to Theorem 4.1,

$$\bar{v} = \tilde{v} \text{ on } [0, T) \times \Omega.$$

Hence, we may chain together the inequalities (5.19) and (5.1) to get

$$2 \leq \sup_{T/2 \leq t < T} \|\tilde{v}(t)\|_{\bar{H}^3} \leq \sup_{0 \leq t < T} \|\bar{v}\|_{\bar{H}^3} \leq 1,$$

which is a contradiction. Therefore, the perturbed problem does not have the global stability of order  $k$  for any  $k \geq 3$ . This completes the proof of Theorem 2.2.

**Acknowledgements.** The research of Fei Jiang was supported by the Fujian Provincial Department of Science and Technology (Grant No. JK2009045) and NSFC (Grant No. 11101044), and the research of Song Jiang by the National Basic Research Program under the Grant 2011CB309705 and NSFC (Grant No. 40890154).

### References

- [1] R.A. Adams, J. John, Sobolev Space, Academic Press: New York, 2005.
- [2] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, The International Series of Monographs on Physics, Oxford, Clarendon Press, 1961.
- [3] R. Duan, F. Jiang, S. Jiang, On the Rayleigh-Taylor instability for incompressible, inviscid magnetohydrodynamic flows, to appear in SIAM J. Appl. Math. (2011).
- [4] D. Erban, Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids, Comm. PDE 13 (1988) 1265–1295.
- [5] D. Erban, The equations of motion of a perfect fluid with free boundary are not well posed, Comm. PDE 12 (1987) 1175–1201.
- [6] Y. Guo, I. Tice, Compressible, inviscid rayleigh-taylor instability, arXiv:0911.4098v2 [math.AP] 23 Feb 2011, to appear in Indiana Univ. Math. J. (2011).
- [7] Y. Guo, I. Tice, Linear rayleigh-taylor instability for viscous, compressible fluids, SIAM J. Math. Anal. 42 (2011) 1688–1720.
- [8] H.J. Hwang, Variational approach to nonlinear gravity-driven instability in a MHD setting, Quart. Appl. Math. 66 (2008) 303–324.
- [9] H.J. Hwang, Y. Guo, On the dynamical Rayleigh-Taylor instability, Arch. Rational Mech. Anal. 167 (2003) 235–253.
- [10] F. Jiang, S. Jiang, W. Wang, On the Rayleigh-Taylor instability for two uniform viscous incompressible flows, submitted (2011).

- [11] M. Kruskal, M. Schwarzschild, Some instabilities of a completely ionized plasma, *Proc. Roy. Soc. (London) A* 233 (1954) 348–360.
- [12] A. Novotný, I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford University Press, USA, 2004.
- [13] L. Rayleigh, Analytic solutions of the Rayleigh equations for linear density profiles, *Proc. London. Math. Soc.* 14 (1883) 170–177.
- [14] L. Rayleigh, Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, *Scientific Paper, II* (1900) 200–207.
- [15] J. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon Press, Oxford, 1965.
- [16] G.I. Taylor, The stability of liquid surface when accelerated in a direction perpendicular to their planes, *Proc. Roy. Soc. A* 201 (1950) 192–196.
- [17] J. Wang, *Two-Dimensional Nonsteady Flows and Shock Waves* (in Chinese), Science Press, Beijing, China, 1994.
- [18] Y.J. Wang, Critical magnetic number in the MHD Rayleigh-Taylor instability, *arXiv:1009.5422v1 [math.AP]* 28 Sep 2010 (2010).
- [19] J. Wehausen, E. Laitone, *Surface Waves*, *Handbuch der Physik* Vol. 9, Part 3, Springer-Verlag, Berlin, 1960.